5

THE SIMPLEX METHOD

5·1 INTRODUCTION

The simplex method is an *iterative* (step-by-step) procedure by which a new *basic feasible solution* can be obtained from a given (initial) basic feasible solution so that the value of the *objective function* is improved. If certain conditions for optimality are justified and the optimality is reached in a finite number of steps, the purpose of this chapter will be complete.

In this chapter, we shall develop the simplex method by establishing the properties of the solutions to the general LPP which, for simplicity, can be easily converted to standard matrix form :

$$\text{Max } z = \mathbf{CX}, \text{ subject to } \mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0},$$

where the system $\mathbf{AX} = \mathbf{b}$ consists of m linear equations in $N \geq m$ decision variables (containing *original, slack* and *surplus* variables) ; and

$$\mathbf{C} = (c_1, c_2, \ldots, c_N), \mathbf{X} = (x_1, x_2, \ldots, x_N), \mathbf{b} = (b_1, b_2, \ldots, b_m)$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mN} \end{bmatrix}.$$

For convenience, column vectors will also be represented by row vectors without using transpose symbol (T). So there should be no confusion in understanding scalar multiplication of two vectors $\mathbf{C}$ and $\mathbf{X}$.

Note. For all theoretical purposes, N will denote the total number of variables after converting the general LPP to above standard form.

For example if l, m, n stand respectively for *slack, surplus* and *original* variables, then $N = l + m + n$.

For derivation of simplex method it is necessary to introduce some notations and definitions in the following section.

Theory of Simplex Method

5·2 SOME DEFINITIONS AND NOTATIONS

Recall the LP problem in standard form :

$$\text{Max } z = \mathbf{CX}, \text{ subject to } \mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}.$$

First, denote the jth column of $m \times N$ matrix $\mathbf{A}$ by $\mathbf{a}_j$, $j = 1, 2, \ldots, N$ so that

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_N] \qquad \ldots(5\cdot1)$$

Now form an $m \times m$ non-singular sub-matrix $\mathbf{B}$ called the *basis matrix,* whose column vectors are m number of linearly independent columns selected from matrix $\mathbf{A}$ given by (3·1) and re-named as $\beta_1, \beta_2, \ldots, \beta_m$. Therefore,

$$\mathbf{B} = [\beta_1, \beta_2, \ldots, \beta_m] \qquad \ldots(5\cdot2)$$

The columns of $\mathbf{B}$ form a basis for R^m.

Now any column $\mathbf{a}_j$ of $\mathbf{A}$ can be expressed as a linear combination of columns of $\mathbf{B}$. Following notation may be used to represent such a linear combination :

$$\mathbf{a}_j = x_{1j}\beta_1 + x_{2j}\beta_2 + \ldots + x_{mj}\beta_m$$
$$= (\beta_1, \beta_2, \ldots, \beta_m)(y_{1j}, y_{2j}, \ldots, y_{mj})$$

or $\quad \mathbf{a}_j = \mathbf{B}\mathbf{x}_j$, where $\mathbf{Y}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$.

or $$x_j = B^{-1} a_j, \quad ...(5.3)$$
where x_{ij} $(i = 1, 2, ...m)$ are the scalars required to express a_j in such a form.

The vectory x_j will change if the columns of A forming B change.

Any basis matrix B will yield a basic solution to $AX = b$. This solution may be denoted by m-component vector as

$$X_B = (x_{B1}, x_{B2}, \ldots, x_{Bm})$$

where X_B is determined from

$$X_B = B^{-1} b. \quad ...(5.4)$$

The subscript i in x_{Bi} shows that the variable x_i corresponds to column β_i of basis matrix B. This, however, does not show which variable of $(AX = b)$ is x_{Bi}. This also recalls that variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ are called *basic variables,* and the remaining n variables are called *non-basic variables.*

Corresponding to any X_B, the m component row vector (C_B) containing the constants taken from the objective function,

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_N x_N,$$

is associated with the basic variables, that is,

$$C_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}).$$

The subscript i shows that c_{Bi} is the coefficient of the basic variable x_{Bi} in the objective function. This notation implies that for any basic feasible solution, since all non-basic variables are zero, the value of the objective function z is given by

$$z = c_{B1} x_{B1} + c_{B2} x_{B2} + \ldots + c_{Bm} x_{Bm} + 0$$

(since all remaining $N - m$ variables are non-basic and hence zero)

$$= (c_{B1}, c_{B2}, \ldots, c_{Bm})\,(x_{B1}, x_{B2}, \ldots, x_{Bm})$$

or $$z = C_B X_B \quad ...(5.5)$$

In linear programming terminology, the constant vector C_B is called *price vector* and this is how one refers them as follows.

Finally, a new variable z_j, is defined as

$$z_j = x_{1j} c_{B1} + x_{2j} c_{B2} + \ldots + x_{mj} c_{Bm} = \sum_{i=1}^{m} c_{Bi} x_{ij} \quad ...(5.6a)$$

$$= (c_{B1}, c_{B2}, \ldots, c_{Bm})\,(x_{1j}, x_{2j}, \ldots, x_{mj})$$

or $$z_j = C_B x_j. \quad ...(5.6b)$$

There exists z_j for each a_j, then z_j corresponding to a_j changes as the columns of A forming B change. This variable z_j will assume special importance in the subsequent analysis.

Above definitions and notations can be clearly understood by the following illustrative numerical example.

5·2-1 An Example to Explain Definitions and Notations

Example 1. *Illustrate definitions and notations by the linear programming problem :*

Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$,

subject to,
$$4x_1 + 2x_2 + x_3 + x_4 = 4$$
$$x_1 + 2x_2 + 3x_3 - x_5 = 8.$$

Solution. First of all constraint equations in matrix form may be written as

$$\overset{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix}}{\begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

or $$AX = b.$$

A basis matrix $\mathbf{B} = (\beta_1, \beta_2)$ is formed using columns $\mathbf{a}_3$ and $\mathbf{a}_1$ so that

$$\beta_1 = \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta_2 = \mathbf{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rank of matrix $\mathbf{A}$ is 2, and hence $\mathbf{a}_3$, $\mathbf{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus, basis matrix is

$$\mathbf{B} = (\beta_1, \beta_2) = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \quad (\text{columns } \mathbf{a}_3, \mathbf{a}_1)$$

Using (5·4), the basic feasible solution is

$$\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b} = \left[\frac{1}{|\mathbf{B}|}\text{adj}(\mathbf{B})\right]\mathbf{b}$$

$$= \frac{-1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or

$$\mathbf{X_B} = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}.$$

Therefore, basic variables are $x_{B1} = 28/11 = x_3$, $x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero) *i.e.*, $x_2 = x_4 = x_5 = 0$. Also,

$$c_{B1} = \text{coefficient of } x_{B1} = \text{coeff. of } x_3 = c_3 = 3$$
$$c_{B2} = \text{coefficient of } x_{B2} = \text{coeff. of } x_1 = c_1 = 1$$

Hence $\mathbf{C_B} = (3, 1)$.

Now, using (5·5), the value of the objective function is

$$z = \mathbf{C_B X_B} = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a}_j = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_2 + x_{22}\beta_1 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5·3) to get

$$\mathbf{x}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $x_{12} = 6/11$, $x_{22} = 4/11$.

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5·6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C_B x_2} = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix}$$

$$\left(3\frac{6}{11} + 1\frac{4}{11}\right) = \frac{22}{11}.$$

Similarly, z_1, z_3, z_4, z_5 can also be computed.

5·3 FUNDAMENTAL PROPERTIES OF SOLUTIONS

Theorem 5·1. ***(Reduction of Feasible Solution to Basic Feasible Solution).*** *If a linear programming problem Max. $z = \mathbf{CX}$ subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq 0$ has at least one feasible solution, then it has at least one basic feasible solution.*

Proof. Let $\mathbf{AX} = \mathbf{b}$ be the linear system in standard form and $\mathbf{A}$ be $m \times N$ matrix. Let $r(\mathbf{A}) = m$. Since there does cexist a feasible solution, we must have $r(\mathbf{A}, \mathbf{b}) = r(\mathbf{A})$ and $m < N$. Consider an arbitrary feasible solution

$$\mathbf{X}^{(0)} = (x_1, x_2, \ldots, x_N), \quad x_j \geq 0. \qquad \ldots(5\cdot7)$$

Also, suppose that the variables have been numbered such that those variables which have *positive* values are first k ones ($k \le N$), the remaining $(N-k)$ variables having the value zero. Thus, (5·7) can be expressed as

$$\mathbf{x}^{(0)} = (x_1, x_2, \ldots, x_k, \overbrace{0, 0 \ldots, 0}^{N-k}). \quad \ldots(5\cdot8)$$

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ be the first k columns of A (associated with the non-zero variables $x_1, x_2, \ldots, x_k$, respectively). Then by hypothesis,

$$x_1\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \ldots + x_k\,\mathbf{a}_k = \mathbf{b} \quad \text{or} \quad \sum_{j=1}^{k} x_j\,\mathbf{a}_j = \mathbf{b}, \quad \ldots(5\cdot9)$$

But, only two possibilities may arise : that is, the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ may be either *linearly independent or dependent.* Now consider these two cases individually.

Case I. When $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are linearly independent ($k \le m$).

Any *feasible solution* for which vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ associated with non-zero variables $x_1, x_2, \ldots, x_k$, respectively, are linearly independent, is called *a basic feasible solution* (by *Corollary* 2). Hence first part of the theorm is true in this case, that is, the feasible solution $(x_1, x_2, \ldots, x_k\,;\, x_{k+1} = 0, \ldots, x_N = 0)$ is, by definition, a basic feasible solution. This solution is *degenerate* if $k < m$, and *non-degenerate* if $k = m$.

Case 2. When $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are linearly dependent ($k > m$).

This is certainly the case when $k > m$. In this case, reduce the number of positive variables (step-by-step) until the columns associated with positive variables become linearly independent. From this feasible solution another feasible solution can be obtained.

Since $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are linearly dependent, then by definition of linear dependence, there exist scalars $\lambda_j, j = 1, 2, \ldots, k$ such that

$$\lambda_1\,\mathbf{a}_1 + \lambda_2\,\mathbf{a}_2 + \ldots + \lambda_k\,\mathbf{a}_k = \mathbf{0} \quad \text{or} \quad \sum_{j=1}^{k} \lambda_j\,\mathbf{a}_j = \mathbf{0} \quad \ldots(5\cdot10)$$

implies the existence of at least one $\lambda_j \neq 0$.

Suppose at least one of the λ_j is positive because, if it were not, we can multiply the equation (5·10) by -1.

Now, let

$$v = \max_{1 \le j \le k} \left(\frac{\lambda_j}{x_j}\right) \quad \ldots(5\cdot11)$$

Obviously, $v > 0$ for $x_j > 0$ ($j = 1, 2, \ldots, k$) and at least one $\lambda_j > 0$.

Now, multiplying the equation (5·10) by $1/v$ and then subtracting from (4·9), we get

$$\sum_{j=1}^{k} x_j\,\mathbf{a}_j - \frac{1}{v}\sum_{j=1}^{k}\left(x_j - \frac{\lambda_j}{v}\right)\mathbf{a}_j = \mathbf{b}, \quad \ldots(5\cdot12)$$

which states that

$$\hat{\mathbf{x}} = \left(x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, \ldots, x_k - \frac{\lambda_k}{v}, \overbrace{0, 0, \ldots, 0}^{N-k}\right)$$

is a *new solution* of the matrix equation $\mathbf{AX} = \mathbf{b}$. Also, from (5·11), we have

$$v \ge \lambda_j/x_j \quad \text{or} \quad x_j \ge \lambda_j/v \quad \text{or} \quad x_j - (\lambda_j/v) \ge 0, j = 1, 2, \ldots, k.$$

Thus, new solution $\hat{\mathbf{x}}$ also satisfies the non-negativity conditions.

Since $x_j - (\lambda_j/v) = 0$ for at least one k, therefore $\hat{\mathbf{x}}$ given by (5·12) is a feasible solution which contains at the most $k - 1$ nonzero variables. All other variables will be zero.

If the columns associated with positive variables are still *linearly dependent,* repeat the whole reduction procedure as described above. Eventually, derive a solution in which columns corresponding to positive variables are linearly independent (none of the $\mathbf{a}_j = \mathbf{0}$ and that a set containing a single non-null vector is always linearly independent).

Thus the theorem is now completely proved.

Corollary. *There exist only a finite number of basic feasible solution.*

Proof. Left as an exercise for the students.

NoteIn order to understand this theorem, numerical examples for reducing a given feasible solution to a basic feasible solution are given in See. 5·3–1

Theroem 5·2. *If a linear programming problem,*

$$\max.\ z = \mathbf{CX}, \text{ such that } \mathbf{AX} = \mathbf{b},\ \mathbf{X} \geq \mathbf{0},$$

has at least an optimal feasible solution, then at least one basic feasible solution must be optimal.

Proof. Let

$$\mathbf{x}^{(0)} = (x_1, x_2, \ldots, x_k, \overbrace{0, 0, \ldots, 0}^{N-k}) \qquad \ldots(5{\cdot}13)$$

be an optimal feasible solution to the given linear programming problem which yields the optimum value

$$z^* = \sum_{j=1}^{k} c_j x_j \qquad \ldots(5{\cdot}14)$$

Now, proceeding as in *Theorem 5·1,* we can reduce $\mathbf{x}^{(0)}$ to a new basic feasible solution

$$\hat{\mathbf{x}} = \left\{x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, \ldots, x_k - \frac{\lambda_k}{v}, \overbrace{0, 0, \ldots, 0}^{N-k}\right\}. \qquad \ldots(5{\cdot}15)$$

Now, we have to prove that $\hat{\mathbf{x}}$ is also an optimum solution.

The new value of the objective function corresponding to new solution $\hat{\mathbf{x}}$ will become

$$\hat{z} = \sum_{j=1}^{k} c_j \left(x_j - \frac{\lambda_j}{v}\right) = \sum_{j=1}^{k} c_j x_j - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j$$

or

$$\hat{z} = z^* - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j \quad \left[\text{since } z^* = \sum_{j=1}^{k} c_j x_j \text{ from eqn. } (5{\cdot}8)\right] \qquad \ldots(5{\cdot}16)$$

For optimality, $\hat{z}$ must be equal to z^*. Hence, $\hat{\mathbf{x}}$ will be optimal solution, if and only if,

$$\sum_{j=1}^{k} c_j \lambda_j = 0 \qquad \ldots(5{\cdot}17)$$

in equation (5·16). This can be proved by contradiction.

If possible, let us suppose that

$$\sum_{j=1}^{k} c_j \lambda_j \neq 0.$$

Then, there will be two possiblities :

(i) $\sum_{j=1}^{k} c_j \lambda_j > 0$, or (ii) $\sum_{j=1}^{k} c_j \lambda_j < 0$.

Now, in either of these two cases a real number (say, r) can be found such that

$$r \sum_{j=1}^{k} c_j \lambda_j > 0,$$

(in the first case, r will be positive, and in the second case r will be negative)

$$\sum_{j=1}^{k} c_j (r\lambda_j) > 0. \qquad \ldots(5{\cdot}18)$$

Now adding $\sum_{j=1}^{k} c_j x_j$ to both sides of (5·18), we get

$$\sum_{j=1}^{k} c_j (r\lambda_j) + \sum_{j=1}^{k} c_j x_j > \sum_{j=1}^{k} c_j x_j$$

or

$$\sum_{j=1}^{k} c_j (x_j + r\lambda_j) > z^*. \qquad \ldots(5{\cdot}19)$$

Now, $(x_1 + r\lambda_1, x_2 + r\lambda_2, \ldots, x_k + r\lambda_k, \overbrace{0, 0, \ldots, 0}^{N-k})$ is also a solution for any value of r which can be observed by multiplying the equation (5·10) by r and adding to equation (5·9).

Furthermore, *there exist an infinite number of choices of r for which the solution*

$$(x_1 + r\lambda_1, x_2 + r\lambda_2, \ldots, x + r\lambda_k, \overbrace{0, 0, \ldots, 0}^{N-k})$$

satisfies the non-negativity restrictions as well.

In order to prove this statement and to satisfy the non-negativity restriction, we need

$$x_j + r\lambda_j \geq 0, j = 1, 2, \ldots, k$$

or
$$r\lambda_j \geq -x_j$$

$$r \geq -\frac{x_j}{\lambda_j}, \text{ if } \lambda_j > 0$$

or
$$r \leq -\frac{x_j}{\lambda_j}, \text{ if } \lambda_j < 0$$

$$r \text{ is unrestricted, if } \lambda_j = 0$$

Thus, if r is selected to satisfy the relationship

$$\max_{\substack{j \\ (\lambda_j > 0)}} \left(-\frac{x_j}{\lambda_j}\right) \leq r \leq \max_{\substack{j \\ (\lambda_j < 0)}} \left(-\frac{x_j}{\lambda_j}\right), \quad \ldots(5{\cdot}20)$$

then $x_j + r\lambda_j \geq 0$ for $j = 1, 2, \ldots, k$. It may also be noted that if there is no j for which $\lambda_j > 0$, then there is no lower limit for r; and if there is no j for which $\lambda_j > 0$, then there is no upper limit for r. Furthermore,

$$\max_{\substack{j \\ (\lambda_j > 0)}} \left(-\frac{x_j}{\lambda_j}\right) < 0 \quad \text{and} \quad \max_{\substack{j \\ (\lambda_j < 0)}} \left(-\frac{x_j}{\lambda_j}\right) > 0.$$

This proves that when r lies in the non-empty interval given in (5·20), then an infinite number of solutions

$$(x_1 + r\lambda_1, x_2 + r\lambda_2, \ldots, x_k + r\lambda_k, \overbrace{0, 0, \ldots, 0}^{N-k})$$

satisfy the non-negativity restrictions as well.

Now, returning to eqn. (5·19), it may be concluded that left hand side $\sum_{j=1}^{k} c_j (x_j + r\lambda_j)$ yields the value of the objective function which is *strictly* greater than the greatest value of objective function, which is not possible. This contradiction proves that (5·19) holds and hence $\hat{\mathbf{X}}$ is optimal.

Alternative Proof of Theorem 5·2 : *By Theoram 4·1, if there exists an optimal solution to a L.P. problem : Max.* $z = \mathbf{CX}$ *subject to* $\mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$, *then there will exist a basic feasible solution.* Hence a basic feasible solution must exist to the given L.P. problem.

Let $z_0 = \mathbf{C_B X_B}$ with $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$, be a basic feasible solution to the problem.

Now to show that the basic feasible solution is optimal, we are required to prove $z_0 \geq z^*$.

The constraint equation $\mathbf{AX} = \mathbf{b}$ can be expressed as

$$\sum_{j=1}^{N} \mathbf{a}_j x_j = \mathbf{b}. \quad \ldots(5{\cdot}21)$$

Since any vector $\mathbf{a}_j \in \mathbf{A}$ can be expressed as a linear combination of vectors in $\mathbf{B}$, *i.e.*

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij} \beta_i \quad \ldots(5{\cdot}22)$$

From (5·21) and (5·22), we have

$$\sum_{j=1}^{N} \left(\sum_{i=1}^{m} x_{ij} \beta_i\right) x_j = \mathbf{b}$$

or
$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\, x_j\right)\beta_i = \mathbf{b}\,.$$

Since $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ is the basic feasible solution to (5·21),

$$\sum_{i=1}^{m} x_{Bi}\,\beta_i = \mathbf{b}\,.$$

Any vector can be uniquely expressed as a function of its basic vectors, therefore

$$\mathbf{x_B} = \sum_{i=1}^{m} x_{Bi}\,\beta_i\,.$$

The optimality condition is $z_j - c_i \ge 0$ and $x_j \ge 0$. Therefore,

$$\sum_{j=1}^{N} x_j z_j \ge \sum_{j=1}^{N} c_j x_j = z^* \qquad ...(5·23)$$

But, we have

$$\sum_{j=1}^{N} x_j z_j \;\ge\; \sum_{j=1}^{N} x_i\left(\sum_{i=1}^{m} c_{Bi}\, y_{ij}\right)$$

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{N} x_j\, y_{ij}\right)c_{Bi} = \sum_{i=1}^{m} x_{Bi}\, c_{Bi} = z_0 \qquad ...(5·24)$$

Using (5·23) in (5·24), we have $z_0 \ge z^*$.

Thus basic feasible solution is optimal.

This completes the proof of the theorem.

Remarks :

1. As already pointed out in *Theorem 5·1*, $\hat{\mathbf{x}}$ has at most $k-1$ non-zero components. Either $\hat{\mathbf{x}}$ is a basic feasible solution or else it is not. In case it is not, we can repeat the procedure as outlined in *Theorems 5·1 & 5·2* to yield another optimal solution $\hat{\mathbf{x}}$ having at the most $k-2$ non-zero components. Continuing in the like manner, an optimal solution is obtained which is basic feasible.
2. It may also be remarked that intentionally a purely algebraic approach for the solution of the linear programming problem has been adopted.
3. It is also important to note that by applying the procedure outlined earlier, we can obtain a basic feasible solution from any feasible solution. Moreover, if the given feasible solution happens to be optimal, the basic feasible solution (obtained by applying the procedure explained earlier) is also optimal.

Q. 1. Prove that if the system $\mathbf{AX} = \mathbf{b}$, of m linear equations in n unknowns ($m \le n$) with rank $(A) = m$ has a feasible solution, then it has a basic feasible solution also.

2. Given a LP problem in standard form, if it has an optimal solution, show that at least one of the basic solutions will be optimal.

3. Show that if the linear progamming problem :

   Max $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \ge 0$,

   has feasible solution, then at least one of the basic feasible solutions will be optimal.

   [**Hint :** Theorems 5·1 & 5·2 combined.]

4. Show that if an L.P.P. has a feasible solution it also has a basic feasible solution.

5. If an L.P.P. Max $z = \mathbf{CX}$ such that $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \ge 0$ where $\mathbf{A}$ is an $m \times n$ matrix of coefficients has at least one feasible solution, then it has at least one basic basic feasible solution also.

6. Give the geometrical equivalent of the statement, "If there is a feasible solution to $\mathbf{AX} = \mathbf{b}$, $r(\mathbf{A}) = m$, then there is a basic feasible solution."

7. Let us consider an L.P.P. having a basic feasible solution. If we drop one of the basis vectors and introduce a non-zero vector in the basis set, then prove that a new solution obtained is also a basic feasible solution.

8. State and prove the fundamental theorem of linear programming.

5·3-1 Numerical Examples

[Reduction of any feasible solution to a basic feasible solution].

Example 2. *If* $x_1 = 2$, $x_2 = 3$, $x_3 = 1$ *be a feasible solution of linear programming problem :*

Max. $z = x_1 + 2x_2 + 4x_3$,

subject to $2x_1 + x_2 + 4x_3 = 11$,

$$3x_1 + x_2 + 5x_3 = 14,$$
$$x_1, x_2, x_3 \geq 0,$$

then find a basic feasible solution from the given feasible solution.

Solution. Step 1. The system may be expressed as

$$\overset{A}{\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}} \overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overset{\mathbf{b}}{\begin{pmatrix} 11 \\ 14 \end{pmatrix}}$$

or $\qquad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$

The given feasible solution is $x_1 = 2, x_2 = 3, x_3 = 1$. Hence,

$$2\mathbf{a}_1 + 3\mathbf{a}_2 + 1\mathbf{a}_3 = \mathbf{b},$$

where $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ 14 \end{bmatrix}.$

Step 2. If the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ associated with the corresponding variables x_1, x_2, x_3 are linearly dependent, then one of these vectors can be expressed as a linear combination of the remaining two. Thus, we have

$$\mathbf{a}_3 = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 \qquad \text{...(5·25)}$$

or $$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or $$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 + \lambda_2 \\ 3\lambda_1 + \lambda_2 \end{bmatrix},$$

which gives us

$$2\lambda_1 + \lambda_2 = 4, \; 3\lambda_1 + \lambda_2 = 5.$$

Solving these two equations, we get $\lambda_1 = 1, \lambda_2 = 2$. Now substituting the values of λ_1 and λ_2 in (5·25), the linear combination is obtained as

$$\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0} \quad \text{or} \quad \sum_{j=1}^{3} \lambda_j \mathbf{a}_j = \mathbf{0} \qquad \text{...(5·26)}$$

where $\qquad \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1.$

Step 3. Now determine which of the three variables (x_1, x_2, x_3) should be zero. For this, find

$$v = \max_{1 \leq j \leq 3} \left(\frac{\lambda_j}{x_j}\right) = \max\left\{\frac{\lambda_1}{x_1}, \frac{\lambda_2}{x_2}, \frac{\lambda_3}{x_3}\right\}$$

or $$v = \max\left\{\frac{1}{2}, \frac{2}{3}, \frac{-1}{1}\right\} = \frac{2}{3}.$$

Since $\left(x_j - \dfrac{\lambda_j}{v}\right) \geq 0,$

$$\hat{\mathbf{x}} = \left(x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, x_3 - \frac{\lambda_3}{v}\right).$$

will be a reduced solution, However,

$$x_1 - \frac{\lambda_1}{v} = 2 - \frac{1}{\frac{2}{3}} = \frac{1}{2}$$

$$x_2 - \frac{\lambda_2}{v} = 3 - \frac{2}{\frac{2}{3}} = 0 \qquad \text{(which was expected also)}$$

$$x_3 - \frac{\lambda_3}{v} = 1 - \left(\frac{-1}{\frac{2}{3}}\right) = 5/2.$$

Step 4. Now this new solution $\hat{\mathbf{x}} = (\frac{1}{2}, 0, \frac{5}{2})$ must be basic feasible if vectors $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ associated with non-zero variables x_1 and x_3 are *linearly independent* (L. I.). Obviously, $\mathbf{a}_1$ and $\mathbf{a}_3$ are L.I.

Hence the required basic feasible solution is

$$x_1 = \tfrac{1}{2}, x_2 = 0, x_3 = \tfrac{5}{2}.$$

To verify this,

$$\frac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \end{bmatrix}.$$

Example 3. *Show that the feasible solution $x_1 = 1, x_2 = 0, x_3 = 1, z = 6$, to the system :*

$$x_1 + x_2 + x_3 = 2$$
$$x_1 - x_2 + x_3 = 0$$
$$2x_1 + 3x_2 + 4x_3 = z \ (Min.)$$

is not basic

Solution. First express the given system of constraint equations in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Therefore, according to usual notations, we have

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Now, we have to show that given feasible solution $x_1 = 1, x_2 = 0, x_3 = 1$ is not basic.

This may be proved on the basis of *Corollary* that *the given feasible solution $x_1 = 1, x_2 = 0, x_3 = 1$ for which vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ associated with non-zero variables x_1 and x_3 respectively, are linearly dependent (not independent) will not be basic.*

Thus, there is only a need to prove that the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are linearly dependent.

Since there exist non-zero scalars $\lambda_1 = 1, \lambda_2 = -1$ such that

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_3 = \mathbf{0}$$

or $$1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is true. Hence, the vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly dependent. Hence the given feasible solution is not *basic*.

Example 4. *Consider the system of equations*

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4.$$

Here $x_1 = 1, x_2 = 1, x_3 = 1$, and $x_4 = 0$ is a feasible solution. Reduce this feasible solution to two different basic feasible solutions.

Solution.

Step 1. First express the given system of equations as

$$\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{matrix} \quad \mathbf{X} \quad \mathbf{b}$$

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 2 & -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad \text{or} \quad \mathbf{AX} = \mathbf{b}$$

or $$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$$

But, the given feasible solution is $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 0$. Therefore,

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{b},$$

where $$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}.$$

Step 2. Since the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ associated with the corresponding variables are *linearly dependent*, one of the vectors can be expessed as a *linear combination* of remaining two.

Thus, we have

$$\mathbf{a}_3 = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 \qquad ...(5\cdot27)$$

or $$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

or $$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2\lambda_1 - \lambda_2 \end{pmatrix}$$

which gives $\lambda_1 + 2\lambda_2 = 4$, $2\lambda_1 - \lambda_2 = 3$.

Solving these two equations we get $\lambda_1 = 2$, $\lambda_2 = 1$. Now substituting these values of λ_1 and λ_2 in equation (5·27), we get the linear combination

$$2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$$

where $$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1.$$

Step 3. Now determine which of the variables x_1, x_2, x_3 should have the value zero. First, compute

$$v = \max_{1 \le j \le 3} \left(\frac{\lambda_j}{x_j}\right) = \max. \left[\frac{\lambda_1}{x_1}, \frac{\lambda_2}{x_2}, \frac{\lambda_3}{x_3}\right]$$

$$= \max. \left[\frac{2}{1}, \frac{1}{1}, \frac{-1}{1}\right] = \frac{2}{1},$$

indicating that new $\hat{x}_1$ should become zero.

Now, since $x_j - \frac{\lambda_j}{v} \ge 0$, we have

$$\hat{\mathbf{x}} = \left(x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, x_3 - \frac{\lambda_3}{v}\right)$$

will be a reduced solution. But,

$$\hat{x}_1 = x_1 - \frac{\lambda_1}{v} = 1 - \frac{2}{2} = 0 \text{ (expected also)}$$

$$\hat{x}_2 = x_2 - \frac{\lambda_2}{v} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$\hat{x}_3 = x_3 - \frac{\lambda_3}{v} = 1 - \left(\frac{-1}{2}\right) = \tfrac{3}{2}.$$

Step 4. Now the new solution $\hat{\mathbf{x}} = (0, \frac{1}{2}, \frac{3}{2}, 0)$ must be basic if the vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ associated with the non-zero variables x_2 and x_3 are linearly independent.

Obviously, $\mathbf{a}_2$ and $\mathbf{a}_3$ are L.I. Hence the required basic feasible solution will be $x_1 = 0, x_2 = \frac{1}{2}, x_3 = \frac{3}{2}, x_4 = 0$.

To determine another basic feasible solution procced exactly on similar lines except consider the linear combination $\mathbf{a}_1 = \lambda_2\mathbf{a}_2 + \lambda_3\mathbf{a}_3$ instead of $\mathbf{a}_3 = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2$ [equation (5·27)]. Value of v will be $\frac{1}{2}$, variable x_3 will become zero. Then, basic feasible solution will come out to be $x_1 = 3, x_2 = 2, x_3 = 0, x_4 = 0$ (where x_3, x_4 are non-basic).

EXAMINATION PROBLEMS

1. Consider the system of equations

$$x_1 + x_2 = 2x_3 = 4$$
$$2x_1 - x_2 + x_3 = 2$$

where $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$ is feasible solution. Is his solution a basic feasible solution ? If not, reduce the given feasible solution to a basic feasible solution.

[Hint. Use *theorem* 5·1 (See Example 2)]

[Ans. Two basic feasible solutions are possible (both degenerate) : (i) [0, 0, 2] with x_1 non-basic, (ii) [0. 0. 2] with x_2 non-basic.]

2. If $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_5 = 5$ be a feasible solutin to the system

$$2x_1 + 2x_2 + x_3 + x_4 = 16$$
$$8x_1 + 4x_2 + 4x_3 + x_5 = 20,$$

then form a basic feasible solution.

[Hint. Use Theorem 5·1 (see example 2)]

3. If $x_1 = 2$, $x_2 = 4$, $x_3 = 1$ be a feasible solution of the linear eqations,

$$2x_1 - x_2 + x_3 = 2$$
$$x_1 + 4x_2 \quad = 18$$
$$x_1, x_2, x_3 \geq 0,$$

then find two basic feasible solutions.

[Ans. (i) $x_1 = 0$, $x_2 = 9/2$, $x_3 = 11/2$(ii) $x_1 = 26/9$, $x_2 = 35/9$, $x_3 = 0$]

4. What do you mean by an optimal basic feasible soltion to LP problem. Is the following solution :

$$x_1 = 1, x_2 = \frac{1}{2}, x_3 = x_4 = x_5 = 0,$$

a basic solution of the equations :

$$x_1 + 2x_2 + x_3 + x_4 \quad = 2$$
$$x_1 + 2x_2 + \frac{1}{2}x_3 \quad + x_5 = 2 \text{ ?}$$

5. If $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$ be a feasible solution of the set of equations :

$$5x_1 + 4(-x_2) + 3x_3 + + x_4 = 3$$
$$2x_1 + \quad x_2 + 5x_3 - 3x_4 = 0$$
$$x_1 + \quad 6x_2 - 4x_3 + 2x_4 = 15$$
$$x_1, x_2, x_3, x_4 \geq 0,$$

then find a B.F.S.

6. [2, 1, 3] is a feasible solution of the set of equations

$$4x_1 + 2x_2 - 3x_3 = 1$$
$$6x_1 + 4x_2 - 5x_3 = 1.$$

Reduce it to a basic feasible solution of the set.

[Ans. $x_1 = 1$, $x_2 = 0$ (non-basic), $x_3 = 1$]

7. Given the linear system

$$2x_1 - x_2 + 2x_3 = 10$$
$$x_1 + 4x_2 \quad = 18$$

and the non-negativity restrictions : $x_1, x_2, x_3 \geq 0$, obtain a basic feasible solution starting from the feasible solution $x_1 = 2$, $x_2 = 4$, $x_3 = 5$.

[Ans. $x_1 = 58/9$, $x_2 = 26/9$, $x_3 = 0$ (non basic)]

8. Consider the set of equations

$$2x_1 - 3x_2 + 4x_3 + 6x_4 \quad = 25$$
$$x_1 + 2x_2 + 3x_3 - 3x_4 + 5x_5 = 12.$$

Note that $x_1 = 2$, $x_2 = 1$, $x_3 = 3$, $x_4 = 2$, $x_5 = 1$ is a feasible solution. Reduce this solution to a basic feasible solution.

[Ans. $x_1 = 3$, $x_2 = 2$, $x_3 = x_4 = 0$ (non basic)]

9. Consider the set of equations

$$5x_1 - 4x_2 = 3x_3 + x_4 = 3$$
$$2x_1 + x_2 + 5x_3 - 3x_4 = 0$$
$$x_1 + 6x_2 - 4x_3 + 2x_4 = 15 .$$

A feasible solution is : $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$. Reduce this solution to a basic feasible solution.

5·4 TO DETERMINE IMPROVED (BETTER) BASIC FEASIBLE SOLUTION

Suppose a basic feasible solution to $\mathbf{AX} = \mathbf{b}$ is given by

$$\mathbf{X_B} = (x_{B1}, x_{B2}, \ldots, x_{Bm}) = \mathbf{B}^{-1}\,\mathbf{b} \qquad \ldots(5{\cdot}28)$$

and gives the value of the objective function

$$z = \mathbf{C_B}\,\mathbf{X_B}$$

To develop a precedure for determining another basic feasible solution which gives a better value of z following theorems are given.

Theorem 5·3. (Replacement of a basis column) *Given a non-degenerate basic feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\,\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$ *which yields a value for the objective function* $z = \mathbf{C_B}\,\mathbf{X_B}$. *If for any column* $\mathbf{a}_j$ *in* $\mathbf{A}$ *but not in* $\mathbf{B}$, $z_j - c_j < 0$ *and if at least one* $x_{ij} > 0$ $(i = 1, 2, \ldots, m)$, *then a new basic feasible solution can be obtained by replacing one of the columns in* $\mathbf{B}$ *by* $\mathbf{a}_j$.

Proof. A new basic feasible solution is obtained by replacing one of the vectors (say $\mathbf{a}_j$) in $\mathbf{A}$ but not in $\mathbf{B}$ by some vectors in $\mathbf{B}$ (say β_r). Therefore,

$$\mathbf{a}_j \neq \beta_i \; (i = 1, 2, \ldots, m).$$

Since $\mathbf{a}_j$ can be expressed as the linear combination of vectors in $\mathbf{B}$, we have

$$\mathbf{a}_j = \sum_{i=1}^{m} x_{ij}\,\beta_i$$

or
$$\mathbf{a}_j = x_{1j}\,\beta_1 + x_{2j}\beta_2 + \ldots + x_{rj}\beta_r + \ldots + x_{mj}\,\beta_m \qquad \ldots(5{\cdot}29)$$

Now, using the replacement theorem $\mathbf{a}_j$ can replace β_r and still maintains the basis matrix, provided $y_{rj} \neq 0$.

Assuming $y_{rj} \neq 0$ from (5·29), $\mathbf{a}_j$ can be written as

$$\mathbf{a}_j = \sum_{i=1, i\neq r}^{m} x_{ij}\,\beta_i + x_{rj}\,\beta_r \qquad \ldots(5{\cdot}30)$$

Solving the equation (5·30) for β_r,

$$\beta_r = \frac{1}{y_{rj}}\,\mathbf{a}_j - \sum_{i=1, i\neq r}^{m} \frac{x_{ij}}{x_{rj}}\,\beta_i \qquad \ldots(5{\cdot}31)$$

Also,

$$\mathbf{BX_B} = \mathbf{b}$$

or
$$(\beta_1, \beta_2, \ldots, \beta_r, \ldots \beta_m)\,(x_{B1}, x_{B2}, \ldots, x_{Br}, \ldots, x_{Bm}) = \mathbf{b}$$

or
$$x_{B1}\,\beta_1 + x_{B2}\,\beta_2 + \ldots + x_{Br}\,\beta_r + \ldots + x_{Bm}\,\beta_m = \mathbf{b}$$

or
$$\sum_{i=1, i\neq r}^{m} x_{Bi}\,\beta_i + x_{Br}\,\beta_r = \mathbf{b} \qquad \ldots(5{\cdot}32)$$

Substituting the value of β_r from (5·31) in (5·32),

$$\sum_{i=1, i\neq r}^{m} x_{Bi}\,\beta_i + x_{Br}\left[\frac{1}{y_{rj}}\,\mathbf{a}_j - \sum_{i=1, i\neq r}^{m} \frac{y_{ij}}{y_{rj}}\,\beta_i\right] = \mathbf{b}$$

or
$$\sum_{i=1, i\neq r}^{m} \left(x_{Bi} - x_{Br}\,\frac{y_{ij}}{y_{rj}}\right)\beta_i + \frac{x_{Br}}{y_{rj}}\,\mathbf{a}_j = \mathbf{b} \qquad \ldots(5{\cdot}33a)$$

or
$$\sum_{i=1, i\neq r}^{m} \hat{x}_{Bi}\,\beta_i + \hat{x}_{Br}\,\mathbf{a}_j = \mathbf{b}, \qquad \ldots(5{\cdot}33b)$$

where $$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}},\ i = 1, 2, \ldots, m\ ;\ i \neq r \qquad \ldots(5\cdot34a)$$

$$\hat{x}_{Br} = \frac{x_{Br}}{y_{rj}} \quad (\text{for } i = r) \qquad \ldots(5\cdot34b)$$

Comparision of (5·33*b*) with (5·32) shows that the new basc solution of $\mathbf{AX} = \mathbf{b}$ becomes

$$\hat{\mathbf{x}}_{\mathbf{B}} = (\hat{x}_{Bi}, \hat{x}_{Br}),\ i = 1, 2, \ldots, m\ ;\ i \neq r$$
$$= (\hat{x}_{B1}, \hat{x}_{B2}, \ldots, \hat{x}_{Br}, \ldots, \hat{x}_{Bm})$$
$$= \left(x_{B1} - x_{Br}\frac{y_{1j}}{y_{rj}},\ x_{B2} - x_{Br}\frac{y_{2j}}{y_{rj}},\ \ldots,\ x_{Bm} - x_{Br}\frac{y_{mj}}{y_{rj}}\right)$$

and other non-basic components are zero.

For the new basic solution to be feasible,

$$\hat{x}_{Bi} \geq 0,\ i = 1, 2, \ldots m,$$

that is, from (5·34*a*) and (5·34*b*)

$$x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \geq 0,\ i = 1, 2, \ldots, m,\ i \neq r \qquad \ldots(5\cdot35a)$$

$$\frac{x_{Br}}{y_{rj}} \geq 0 \qquad \ldots(5\cdot35b)$$

From (5·35*a*), $y_{rj} > 0$ since we start with a non-degenerate basic feasible solution, $x_{Bi} > 0,\ (i = 1, 2, \ldots, m)$. If $y_{rj} > 0$, and $y_{ij} \leq 0\ (i \neq r)$, then (5·35) is satisfied. If $y_{rj} > 0$ and $y_{ij} > 0\ (i \neq r)$, then equation (5·35*a*) is satisfied only when

$$\frac{x_{Bi}}{x_{ij}} - \frac{x_{Br}}{x_{rj}} \geq 0 \qquad [\text{dividing (5·35}a\text{) by } y_{ij} > 0] \qquad \ldots(5\cdot36)$$

or $$-\frac{x_{Br}}{y_{rj}} \geq -\frac{x_{Bi}}{y_{ij}}$$

or $$\frac{x_{Bi}}{y_{rj}} \leq \frac{x_{Bi}}{y_{ij}}$$

or $$\frac{x_{Br}}{y_{rj}} = \min_i \left[\frac{x_{Bi}}{y_{ij}}\right].$$

Thus, if r is selected such that

$$v = \frac{x_{Br}}{y_{rj}} = \min_i \left[\frac{x_{Bi}}{y_{ij}},\ y_{ij} > 0\right], \qquad \ldots(5\cdot37)$$

then column β_r will be removed from the basis matrix $\mathbf{B}$ to replace $\mathbf{a}_j$ so that the new basic solution will be feasible.

Hence the theorem is proved.

Remarks :

1. A new non-singular matrix obtained from **B** by replacing β_r with $\mathbf{a}_j$, is denoted by
$$\hat{\mathbf{B}} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_m),$$
where $\hat{\beta}_i = \beta_i,\ i \neq r$, and $\hat{\beta}_r = \mathbf{a}_j$. ...(5·38)
2. If the minimum in (4·37) is not unique, the new basic solution will be degenerate. Because, in this case, the number of positive basic variables will become less than m.

The procedure developed in this theorem can be explained by the following example.

Q. How would you proceed to change the basic feasible solution in case it is not optimal.

5·4-1 Numerical Example

Example 5. *Given the non-degenerate basic feasible solution* $x_3 = 4$ and $x_4 = 8$ *to the following LP problem :*

$$Max. \quad z = x_1 + 2x_2,$$

subject to

$$x_1 + 2x_2 + x_3 = 4,$$
$$x_1 + 4x_2 + x_4 = 8.$$

Obtain the new basic feasible solution.

Solution. The given basic feasible solution can be expressed as

$$\mathbf{B}\mathbf{X}_B = \mathbf{b}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

Here, $\mathbf{X}_B = \begin{pmatrix} x_{B1} \\ x_{B2} \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} \overset{\mathbf{a}_1}{1} & \overset{\mathbf{a}_2}{2} & \overset{\beta_1}{1} & \overset{\beta_2}{0} \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 8 \end{pmatrix}$$

where x_j's for every column $\mathbf{a}_j$ in $\mathbf{A}$ but not in $\mathbf{B}$ are

$$\mathbf{x}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$$

$$\mathbf{x}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

Since $x_{11} = 1$, $x_{21} = 1$ which we greater than zero, we can insert $\mathbf{a}_1$ in $\mathbf{B}$. Select β_r for replacement by $\mathbf{a}_1$ which corresponds to the value of suffix r determined by the *minimum ratio rule* :

$$\frac{x_{Br}}{x_{r1}} = \min_i \left[\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right]$$

Therefore,

$$\frac{x_{Br}}{x_{r1}} = \min \left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right]$$

$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{4}{1}, \frac{8}{1}\right] = \frac{4}{1}$$

$$\frac{x_{Br}}{x_{r1}} = \frac{x_{B1}}{x_{11}}.$$

Equating the suffices on both sides, $r = 1$. Hence, remove β_r (for $r = 1$), that is β_1.

The new basis matrix becomes

$$\hat{\mathbf{B}} = (\hat{\beta}_1, \hat{\beta}_2) = (\mathbf{a}_1, \beta_2) \qquad \text{(because } \mathbf{a}_1 \text{ is replaced by } \beta_1)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The new basic feasible solution $\hat{\mathbf{X}}_B$ can be obtained either by using the result $\hat{\mathbf{X}}_B = \hat{\mathbf{B}}^{-1}\mathbf{b}$ or using the transformation formulae (5·34*a*) and (5·34*b*) directly. Using (5·34*a*) and (5·34*b*) to obtain the new basic feasible solution

$$\hat{x}_{B1} = \frac{x_{B1}}{x_{11}} = \frac{4}{1} = 4$$

$$\hat{x}_{B2} = x_{B2} - x_{B1}\frac{x_{21}}{x_{11}} = 8 - 4 \times \frac{1}{1} = 4$$

The solution to the original system of equations becomes

$$x_1 = x_{B1} = 4, x_2 = 0, x_3 = 0, x_4 = x_{B2} = 4.$$

It is important to note that, if it was decided to insert $\mathbf{a}_2$ in stead of $\mathbf{a}_1$, the new basic feasible solution would have been degenerate (students are advised to verify this point). We have developed so far the procedure for obtaining a new basic feasible solution. Now we must determine whether the value of the objective function corrsponding to this new basic feasible solution has improved, that is, whether $\hat{z} > z$, where $\hat{z}$ denotes the new value of the objective function. For this, we prove the following theorem.

Theorem 5·4. (Improved BFS). *Assume that a non-degenerate basic feasible solution* $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$ *which yields a value for the objective function* $z = \mathbf{c_B}\,\mathbf{x_B}$. *Further suppose that a new basic feasible solution* $\hat{\mathbf{x}} = \hat{\mathbf{B}}^{-1}\mathbf{b}$ to $\mathbf{AX} = \mathbf{b}$ *obtained by replacing one of the columns in* $\mathbf{B}$ *by a column* $\mathbf{a}_j$ *(for which* $x_{ij} > 0$*) in* $\mathbf{A}$ *but not in* $\mathbf{B}$. *Then, if* $z_j - c_j < 0$, *the new vlue of the objective function (denoted by* $\hat{z}$*) will be greater than* z; *i.e.,* $\hat{z} > z$.

Proof. The value of the objective function for the original basic feasible solution is

$$z = \mathbf{c_B}\,\mathbf{x_B}$$
$$= (c_{B1}, c_{B2}, \ldots, c_{Bm})\,(x_{B1}, x_{B2}, \ldots, x_{Bm})$$

or $$z = \sum_{i=1}^{m} c_{Bi}\,x_{Bi}. \qquad \ldots(5\cdot39)$$

The new value is,

$$\hat{z} = \hat{\mathbf{c}}_\mathbf{B}\,\hat{\mathbf{x}}_\mathbf{B}$$

or $$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi}\,\hat{x}_{Bi} = \sum_{i=1,\, i\neq r}^{m} \hat{c}_{Bi} + \hat{c}_{Br}\,\hat{x}_{Br}$$

where $\hat{c}_{Bi} = c_{Bi}\ (i \neq r)$ and $\hat{c}_{Br} = c_j$.

Therefore, $$\hat{z} = \sum_{i=1,\, i\neq r}^{m} c_{Bi}\,\hat{x}_{Bi} + c_j\,\hat{x}_{Br}.$$

Substituting the values of new variables $\hat{x}_{Bi}$ and $\hat{x}_{Br}$ from (5·34*a*) and (5·34*b*) into the last expression, we get

$$\hat{z} = \sum_{i=1,\, i\neq r}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}\right) + c_j\,\frac{x_{Bj}}{x_{rj}} \qquad \ldots(5\cdot40)$$

Since the term for which $i = r$ is

$$c_{Br}\left(x_{Br} - x_{Br}\frac{x_{rj}}{x_{rj}}\right) = 0,$$

we can include it in the summation (5·40) without changing the value of $\hat{z}$, so that

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{x_{ij}}{x_{rj}}\right) + c_j\,\frac{x_{Br}}{x_{ri}}$$
$$= \sum_{i=1}^{m} c_{Bi}x_{Bi} - \frac{x_{Br}}{x_{rj}}\sum_{i=1}^{m} c_{Br}\,y_{ij} + \frac{x_{Br}}{x_{rj}}\,c_j$$
$$= z - \frac{x_{Br}}{x_{rj}}\,z_j - \frac{x_{Br}}{x_{rj}}\,c_j = z - (z_j - c_j)\,\frac{x_{Br}}{x_{rj}}$$

or $$\hat{z} = z - (z_j - c_j)\,v, \qquad \ldots(5\cdot41)$$

where $v = x_{Br}/x_{rj}$.

Now, from (5·41) it is seen that the new value $\hat{z}$ of the objective function is the original value z minus the quantity $(z_j - c_j)\,v$. If, $\hat{z}$ is to exceed z, then the quantity $(z_j - c_j)\,v$ must be less than zero. Since $v > 0$, $z_j - c_j$ must be less than zero, that is, if $z_j - c_j < 0$, the value of the objective function is improved, and thus the theorem is proved.

5·4-2 Numerical Example

Example 6. *In Example 5, show that the new value of the objective function is improved.*

Solution. Since $c_1 = 4$, $c_2 = 4$, $c_3 = 0$, $c_4 = 0$, then the original solution $x_3 = 4$, $x_4 = 8$, $x_1 = x_2 = 0$ gives

$$z = 1 \times 0 + 2 \times 0 + 0 \times 4 + 0 \times 8 = 0.$$

In the new basic feasible solutin x_1 replaces x_3. Since

$$z_1 = \mathbf{C_B Y_1} = (0, 0)\begin{pmatrix}1\\1\end{pmatrix} = 0$$

and $z_1 - c_1 = 0 - 1 < 0$, $\hat{z}$ must exceed z (= 0). From (4·41),

$$\hat{z} = 0 - 4\,(0 - 1) = 4 > z\,(= 0).$$

5·4-3 Basic Feasible Solution Increasing z to Maximum Possible Extent

So far a procedure has been developed to get a new basic feasible solution which simply improves the value of the objective function z. Arbitrarily selected $\mathbf{a}_j$ (not in B) is to be replaced by β_r in order to get the improved basic feasible solution. At every step of simplex method, our aim is to get such improved basic feasible solution which gives the greatest increase in z. This can be done by selecting among all $\mathbf{a}_j$ (not in B) such vector $\mathbf{a}_k$ to be replaced by β_r which will increase z maximum possible. So a criterion for selecting the vector $\mathbf{a}_k$ is given below in *Theorem 5·5*.

Theorem 5·5. *If the vector $\mathbf{a}_k$ to be replaced by β_r, the suffix k can be pre-decided by means of the rule :*

$$z_k - c_k = \min\,[z_j - c_j],\ z_j - c_j < 0.$$

Then the value of z is increased as much as possible for the new basic feasible solution.

Proof. In *Theorem 5·4*, the improved value of z is obtained as

$$\hat{z} = z - \frac{x_{Br}}{x_{rj}}\,(z_j - c_j)$$

Thus in order to get maximum value of $\hat{z}$, select that value of j (say k) for which the term $\frac{x_{Br}}{x_{rj}}\,(z_j - c_j)$ is minimum.

The computational difficulty arises while obtaining min $\frac{x_{Br}}{x_{rj}}\,(z_j - c_j)$, because it is necessary to compute $\frac{x_{Br}}{y_{rj}}$ for each $\mathbf{a}_j$ having $z_j - c_j < 0$ by the formula :

$$\frac{x_{Br}}{x_{rj}} = \min_{j}\left[\frac{x_{Bi}}{x_{ij}},\ x_{ij} > 0\right]$$

The change in objective function depends on x_{Br}/x_{rj} and $z_j - c_j$

Thus to avoid large number of computations of x_{Br}/x_{rj} we can neglect the consideration of x_{Br}/x_{rj}.

Hence the most convenient and time saving device for choosing the vector $\mathbf{a}_k$ to enter the basis B consists of selecting the smallest $z_j - c_j$. This is equivalent to choosing the vector $\mathbf{a}_j$ to replace β_r by means of

$$z_k - c_k = \min_{j}\,(z_j - c_j),\ \text{for}\ z_j - c_j < 0. \qquad \text{...(5·42)}$$

Thus the theorem is proved.

Advantages of using the criterion (5·42) :

The following are the advantages of using the criterion (5·42) :

1. The choice of vector $\mathbf{a}_k$ to enter the basis B by using (5·42) gives the greatest possible increase in z at each step.
2. Once a vector $\mathbf{a}_k$ had been inserted into B, it would never have to be removed again, although no criteria has been developed to guarantee this fact.
3. More that m iterations will not be needed to reach the optimal basic feasible solution..
4. It saves considerable time by giving the required solution in least number of steps.

5·4-4 Implications of Degenerate Optimal Basic Feasible Solution

Upto this stage the original basic feasible solution was assumed non-degenerate. Now it remains to examine the implications of *degenerate* original basic feasible solutions.

Theorem 5·6. *Given a degenerate basic feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$. *If for any column* $\mathbf{a}_j$ *in* A *but not in* B, *at least one* $y_{ij} > 0$ $(i = 1, 2, \ldots, m)$, *then a new basic feasible solution is obtained which may or may not be degenerate, by replacing one of the columns, in* B *by* $\mathbf{a}_j$.

Proof. Follow the proof of the *Theorem 5·3* (page 125) up to equation (5·37) on page 127. If some of the $y_{ij} > 0$ correspond to $x_{Bi} = 0$, then $v = 0 = x_{Br}$ and a basic feasible solution will exist and it will be degenerate. Since $\mathbf{X_{B\,r}} = 0$, (5·34*a*) and (5·34*b*) show that the values of the variables which have been retained in the new solution are uncharnged $(\hat{x}_{Bi} = x_{Bi}, i \neq r)$. On the other hand, if all $x_{ij} > 0$ correspond to $x_{Bi} > 0$, none of the $x_{Bi} = 0$ enter into (5·37), so that $v > 0$ and, from [(5·34*a*) and (5·34*b*)], the new solution will be non-degenerate, provided that

$$\min.\left\{\frac{x_{Bi}}{y_{ij}}, y_{ij} > 0\right\} \text{ is unique.}$$

Theorem 5·7. *Suppose a degenerate basic feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$. *If for any column* $\mathbf{a}_j$ *in* A *but not in* B, *at least one* $x_{ij} \neq 0$ $(i = 1, 2, \ldots, m)$ *and if the corresponding* $x_{Bi} = 0$, *then a new degenerate basic feasible solution can be obtained.*

Proof. Follow the proof of the *Theorem 5·3* up to equation (5·35). If $x_{Br} = 0$, then as long as $y_{rj} \neq 0$, equation (5·35) is satisfied and $\mathbf{a}_j$ can be substituted for β_r to yield a new degenerate (since $x_{Br}/y_{rj} = 0$) basic feasible solution in which $\hat{x}_{Bi} = x_{Bi} = (i \neq r)$.

Theorem 5·8. *Given a degenerate basic feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$ *which gives a value for the objective function of* $z = \mathbf{C_B}\,\mathbf{X_B}$. *Also, a new basic feasible solution is obtained with* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$ *by replacing one of the columns in* B *by a column* $\mathbf{a}_j$ *(for which some* $x_{ij} > 0$*) in* A *but not in* B. *Then, if* $z_j - c_j \leq 0$, *the new value of the objective function will be such that* $\hat{z} \geq z$.

Proof. Follow the proof of the *Theorem 5·4* up to equation (5·41). From *Theorem 5·6* we know that $v\,(z_j - c_j) \leq 0$ (since $v > 0$ and $z_j - c_j \leq 0$), implying that $\hat{z} \geq z$.

5·4-5 Summary of Theorems (5·3-5·8)

Now it will prove to be very useful if we could concentrate on the essence of above six theorems.

To sum up this section, it is proved so far that given a basic (degenerate or non-degenerate) feasible solution $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ to $\mathbf{AX} = \mathbf{b}$ which gives a value for the objective function $z = \mathbf{C_B}\,\mathbf{X_B}$ if for any column $\mathbf{a}_j$ in A but not in B, $z_j - c_j < 0$ and if at least one $y_{ij} > 0$ $(i = 1, 2, \ldots, m)$ then a new basic feasible solution can be obtained by replacing one of the columns in B by $\mathbf{a}_j$ and the new value $(\hat{z})$ of the objective function will be such that $\hat{z} \geq z$.

Furthermore, to bring the maximum increase in z, we should select a particular vector $\mathbf{a}_k$ from all $\mathbf{a}_j$ (in A but not in B) to enter the basis matrix B for which the suffix k is pre-determined by using the formula $z_k - c_k = \min_j\,(z_j - c_j)$.

5·5 CONDITIONS FOR LP PROBLEM TO POSSESS UNBOUNDED SOLUTION

In theorems given earlier, it has been proved that for $\mathbf{a}_j$ inserted not in the basis matrix B there is at least one $x_{ij} > 0, i = 1, 2, \ldots, m$.

Now, the important point is : what will be the implication if for at least one $\mathbf{a}_j$ all $x_{ij} \leq 0$. This has been proved in the following theorem.

Theorem 5·9. (Unbounded Solution). *Given any feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$. *If for this solution there is some column* $\mathbf{a}_j$ *in* A *but not in* B *for which* $z_j - c_j < 0$ *and* $x_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$ *then, if the objective function is to be maximized, the problem has an unbounded solution.*

Proof. Insert $\mathbf{a}_j$ in B. It is given that the vector $\mathbf{a_j}$ has all $x_{ij} \leq 0$. Since $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ is the basic feasible solution to $\mathbf{AX} = \mathbf{b}$, therefore from equation (5·32), we have

$$\sum_{i=1}^{m} x_{Bi}\,\beta_i = \mathbf{b} \qquad \ldots(5·43)$$

The value of the objective function is

$$z = \mathbf{C_B X_B} = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} \qquad ...(5\cdot44)$$

Let λ be any scalar. If $\lambda \mathbf{a}_j$ be added and subtracted in (5·43), then we get

$$\sum_{i=1}^{m} x_{Bi}\, \beta_i - \lambda \mathbf{a}_j + \lambda \mathbf{a}_j = \mathbf{b} \qquad ...(5\cdot45)$$

But, from (5·29), we have

$$\mathbf{a}_j = \sum_{i=1}^{m} x_{ij}\, \beta_j,$$

or

$$-\lambda \mathbf{a_j} = -\lambda \sum_{i=1}^{m} x_{ij}\, \beta_i \qquad ...(5\cdot46)$$

Now substituting the value of $-\lambda \mathbf{a_j}$ from (5·46) in (5·45),

$$\sum_{i=1}^{m} x_{Bi}\, \beta_i - \lambda \sum_{i=1}^{m} y_{ij}\, \beta_i + \lambda \mathbf{a}_j = \mathbf{b}$$

or

$$\sum_{i=1}^{m} (x_{Bi} - \lambda y_{ij})\, \beta_i + \lambda \mathbf{a}_j = \mathbf{b} \qquad ...(5\cdot47\ a)$$

Thus, (5·47 *a*) gives the new solution whose variables are

$$\hat{x}_{Bi} = x_{Bi} - \lambda y_{ij},\ i = 1, 2, \ldots, m$$
$$\hat{x}_{m+1} = \lambda \qquad ...(5\cdot47\ b)$$

Since all $y_{ij} \le 0$, when $\lambda > 0$, $x_{Bi} - \lambda y_{ij} \ge 0$, so that (5·47 *b*) gives a feasible solution in which the number of positive variables are less than or equal to $m + 1$ (less than $m + 1$ because some $x_{Bi} - \lambda y_{ij}$ may be zero). In case, the number of positive variables in (5·47 *b*) is equal to $m + 1$ (which is greater than m, the number of constraint equations), the solution (5·47 *b*) will be non-basic feasible solution.

Now, the new value (denoted by $\hat{z}$) of the objective function corresponding to new solution (5·47 *b*) becomes

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}\,(x_{Bi} - \lambda y_{ij}) + c_j \lambda$$

or

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} - \lambda \left(\sum_{i=1}^{m} c_{Bi} - c_j \right)$$

or

$$\hat{z} = z - \lambda\,(z_j - c_j) \qquad \text{[from (5·44) and (5·27 b)]} \qquad ...(5\cdot48)$$

Since it is given that $z_j - c_j < 0$, the value of $\hat{z}$ [given by (5·48)] can be made as large as we please by giving λ a sufficiently large value. But, by definition, a linear programming problem has an **unbounded solution** if the value of the objective function can be no finite maximum value of z. Hence the theorem is proved.

5·5-1 Numerical Example

Example 7. *Show that the linear programming problem*

$$Max\ z = x_1 + 2x_2 + 0x_3 + 0x_4$$

subject to

$$x_1 - x_2 + x_3 = 4$$
$$x_1 - 5x_2 + x_4 = 8$$

has an unbounded solution.

Solution. According to usual notations,

$$c_1 = 1,\ c_2 = 2,\ c_3 = 0,\ c_4 = 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -5 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

If the basis matrix is

$$\mathbf{B} = \left(\beta_1, \beta_2 \right) = \left(\mathbf{a}_1, \mathbf{a}_4 \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then the basic solution $\mathbf{X_B}$ is obtained from

$$\mathbf{BX_B} = \mathbf{b}$$

or
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{B1} \\ x_{B2} \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

or
$$\begin{pmatrix} x_{B1} \\ x_{B1} + x_{B2} \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

Hence, $x_{B1} = 4, x_{B2} = 4$ or $\mathbf{X_B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Then, $\mathbf{x}_2 = \mathbf{B}^{-1} \mathbf{a}_2$ or $\mathbf{Bx}_2 = \mathbf{a}_2$

or
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

or $x_{12} = -1, x_{22} = -4$, that is, both are negative.

Also,
$$z_2 = c_{B1} x_{12} + c_{B2} x_{22} = (1)(-1) + (0)(-4) = -1$$
(since $c_{B1} = c_1 = 1, c_{B2} = c_4 = 0$)

$$z_2 - c_2 = -1 - 2 < 0$$

Now, using the result (5·48),

$$\hat{z} = z - \lambda (z_2 - c_2) = z + 3\lambda$$

If λ is sufficiently increased, $\hat{z}$ can be made arbitrarily large. Thus an *unbounded solution* exists.

5·6 CONDITION WHEN IMPROVED BASIC FEASIBLE SOLUTION BECOMES OPTIMAL (OPTIMALITY CRITERION)

In the absence of degeneracy, we have proved (in *Theorems 5·3 and 5·4*) that, *if for any column* $\mathbf{a}_j$ *in* **A** *but not in* **B** *we have* $z_j - c_j < 0$, *and if at least one* $x_{ij} > 0$ $(i = 1, 2, \ldots, m)$, *then we can improve the current basic feasible solution such that the new value of the objective function will be greater than z, i.e.* $\hat{z} > z$.

But this process of improving a given basic feasible solution cannot be continued indefinitely becasue, there exists only a finite number of basic feasible solutions. Also, in the absence of degeneracy, no basis matrix can ever be repeated because the new value of z increses at each round of improvement, and the same basis matrix cannot give two different values of z.

From above, we conclude that the process of improvement in z cannot be continued further as soon as we reach one of the following two possibilities :

(i) $z_j - c_j < 0$, for at least one j for which $\mathbf{a}_j$ is not in the basis matrix **B** and corresponing to this j, all $y_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$

(ii) $z_j - c_j \geq 0$ for all j for which $\mathbf{a}_j$ in **A** but not in **B**.

During the process of improving the value of z, if we reach the condition (i) at any step we get an unbounded solution (as already proved in *Theorem 5·9*).

Now we hope that we will get an optimal solution if our process of improving the value of z terminates with condition (ii). For this, a theorem for optimality condition has been dealt with.

Theorem 5·10. (Optimality Conditions). *Suppose a basic (degenerate or non-degenerate feasible solution* $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ *to* $\mathbf{AX} = \mathbf{b}$ *with* $z^* = \mathbf{C_B X_B}$ *is obtained at any iteration of simplex method. If* $z_j - c_j \geq 0$ *for every column* $\mathbf{a}_j$ *in* **A** *but not in* **B**, *then* z^* *is the optimum value of the objective function* $z = \mathbf{CX}$ *and* $\mathbf{X_B}$ *is an optimal basic feasible solution.*

Proof. Given that at any iteration of simplex method, the basic feasible solution is

$$\mathbf{X_B} = (x_{B1}, x_{B2}, \ldots, x_{Bm}) = \mathbf{B}^{-1}\mathbf{b}.$$

The basis matrix is $\mathbf{B} = (\beta_1, \beta_2, \ldots, \beta_m)$, and the value of the objective function for this solution is

$$z^* = \mathbf{C_B X_B} = \sum_{i=1}^{m} c_{Bi} x_{Bi}. \qquad \text{...(5·49)}$$

Also, at this stage, we are faced with the given situation $z_j - c_j \geq 0$ for those j for which $\mathbf{a}_j$ is not in $\mathbf{B}$. Let the value of the objective function be

$$z = \mathbf{CX} = \sum_{j=1}^{N} c_j x_j \qquad \text{...(5·50)}$$

for any feasible solution $\mathbf{X} = (x_1, x_2, \ldots, x_N)$ of $\mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$.

Now in order to prove $z^* \geq z$ in this theorem, following steps are necessary :

$$\begin{aligned} \mathbf{X_B} &= \mathbf{B}^{-1}\mathbf{b} = \mathbf{B}^{-1}(\mathbf{AX}) \qquad \text{(because } \mathbf{AX} = \mathbf{b}\text{)} \\ &= (\mathbf{B}^{-1}\mathbf{A})\mathbf{X} \\ &= \mathbf{B}^{-1}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N)\mathbf{X} \\ &= (\mathbf{B}^{-1}\mathbf{a}_1, \mathbf{B}^{-1}\mathbf{a}_2, \ldots, \mathbf{B}^{-1}\mathbf{a}_N) \cdot (x_1, x_2, \ldots, x_N) \\ &= (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)(x_1, x_2, \ldots, x_N) \\ &= \begin{bmatrix} x_{11} & x_{12} \cdots x_{1j} & \cdots & x_{1N} \\ x_{21} & x_{22} \cdots x_{2j} & \cdots & x_{2N} \\ \vdots & \vdots \quad \vdots & & \vdots \\ x_{m1} & x_{m2} \cdots x_{mj} & \cdots & x_{mN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{bmatrix} \end{aligned}$$

or

$$(x_{B1}, \ldots, x_{Bi}, \ldots, x_{Bm}) = \left(\sum_{j=1}^{N} x_{1j} x_j, \sum_{j=1}^{N} x_{2j} x_j, \ldots, \sum_{j=1}^{N} x_{ij} x_j, \ldots, \sum_{j=1}^{N} x_{mj} x_j \right)$$

(*multiplying the matrices on the right side*)

Now, equating ith component on both sides, we get

$$x_{Bi} = \sum_{j=1}^{N} x_{ij} x_j. \qquad \text{...(5·51)}$$

It is given that $z_j - c_j \geq 0$ for all j for which $\mathbf{a}_j$ is not in $\mathbf{B}$. If it is possible to prove that this inequality also holds for all those j for which $\mathbf{a}_j$ is in $\mathbf{B}$, then $z_j - c_j \geq 0$ holds for all j for which $\mathbf{a}_j$ may also be in the basis matrix $\mathbf{B}$. To show this, consider $\beta_i = \mathbf{a}_j$, then

$$\mathbf{x}_j = \mathbf{B}^{-1}\mathbf{a}_j = \mathbf{B}^{-1}\beta_i = \mathbf{e}_i.$$

($\mathbf{e}_i$ denotes the unit vector whose ith component is unity)

For such j,

$$\begin{aligned} z_j &= \mathbf{C_B x}_j \qquad \text{[from (5·6 } b\text{)]} \\ &= \mathbf{C_B e}_i \\ &= (c_{B1}, c_{B2}, \ldots, c_{Bi}, \ldots, c_{Bm})(0, 0, \ldots, \ldots, 1, \ldots, 0) \end{aligned}$$

ith component

$$= c_{Bi} = c_j \qquad \text{(because } \beta_i = \mathbf{a}_j\text{)}$$

Thus, $z_j - c_j = 0$ for those j for which $\mathbf{a}_j$ is in the basis $\mathbf{B}$.

Hence $z_j - c_j \geq 0$, for all $j = 1, 2, \ldots, N$

or $c_j \leq z_j$.

Now multiplying both sides by x_j,

$$c_j x_j \leq z_j x_j \qquad \text{(since } x_j \geq 0 \text{ is feasible solution)}$$

or

$$\sum_{j=1}^{N} c_j x_j \leq \sum_{j=1}^{N} z_j x_j$$

or $\quad z \le \sum_{j=1}^{N} x_j \; (\mathbf{C_B x_j})$ [using (5·50)]

or $\quad z \le \sum_{j=1}^{N} x_j \left(\sum_{i=1}^{m} c_{Bi}\, x_{ij} \right)$ [using (5·6 *a*)]

or $\quad z \le \sum_{i=1}^{m} c_{Bi} \left[\sum_{j=1}^{N} x_j\, x_{ij} \right]$

or $\quad z \le \sum_{i=1}^{m} c_{Bi}\, \lambda_{Bi}$ [using (5·51)]

or $\quad z \le z^*$ [using (5·49)]

which was to be proved. Thus, the theorem is now completely established.

Note. One important point to be noted here is that this proof did not require x_j to be non-degenerate, so the theorem holds for both degenerate and non-degenerate solutions.

Q 1. Given a basic feasible solution to a linear programming problem, show how we can improve this basic feasible solution. State the conditions for optimality.
[Hint. See Theorems 5·3, 5·4 & 5·10].

2. Given a general linear programming problem, explain how you would test whether a basic feasible solution is an optimal solution or not.

3. If for a basic feasible solution $\mathbf{X} = (\mathbf{X_B}, \mathbf{0})$ for a given problem min. $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \ge 0$ it is true that $z_j - c_j \ge 0$ for all j, then prove that the solution is optimum.

5·6-1 A Numerical Example to Demonstrate the Optimality Condition

Example 8. *Show that* $x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 4$ *is the optimal basic feasible solution to the linear programming problem*:

Maximize $\quad z = x_1 + 2x_2 + 0x_3 + 0x_4,$

Subject to $\quad x_1 + 2x_2 + x_3 = 4$

$x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3 x_4 \ge 0.$

Solution. Consider the basic feasible solution $\mathbf{BX_B} = \mathbf{b}$ to $\mathbf{AX} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \mathbf{B} = (\mathbf{a}_1, \mathbf{a}_4) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

$$c_1 = 1, c_2 = 2, c_3 = 0, c_4 = 0, \mathbf{X_B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{C_B} = (c_{B1}, c_{B2}) = (1, 0)$$

Since $\mathbf{a}_2$ and $\mathbf{a}_3$ are not in $\mathbf{B}$, compute $z_2 - c_2$ and $z_3 - c_3$. As $z_2 = \mathbf{C_B Y_2}$ and $z_3 = \mathbf{C_B x_3}$, so first compute $\mathbf{x}_2$ and $\mathbf{x}_3$ and then z_2 and z_3.

Now, $\mathbf{x}_2 = \mathbf{B}^{-1}\mathbf{a}_2$ or $\mathbf{B\,x}_2 = \mathbf{a}_2$. Hence

$$\therefore \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} x_{12} \\ x_{12} + x_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

or $\quad x_{12} = 2, x_{22} = 2.$

Similarly,

$$\mathbf{x}_3 = \mathbf{B}^{-1}\mathbf{a}_3 \text{ or } \mathbf{Bx}_3 = \mathbf{a}_3, \text{ that is, } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or $\quad x_{13} = 1, x_{23} = -1$

Now compute,

$$z_2 = \mathbf{C_B x_2} = c_{B1}\, x_{12} + c_{B2}\, x_{22} = (1)(2) + (0)(2) = 2$$

$$z_3 = \mathbf{C_B x_3} = c_{B1}\, x_{13} + c_{B2}\, x_{23} = (1)(1) + (0)(-1) = 1$$

Therefore,

$$z_2 - c_2 = 2 - 2 = 0$$
$$z_3 - c_3 = 1 - 0 = 1$$

satisfying optimality conditions $z_j - c_j \geq 0$ for all j for which $\mathbf{a}_j$ is not in $\mathbf{B}$, that is, for $j = 2, 3$. Hence the given basic feasible solution is optimal. The maximum value of the objective function is

$$z^* = \mathbf{C_B X_B} = (1, 0)(4, 4) = (1)(4) + (0)(4) = 4.$$

To verfy that $z_j - c_j = 0$ for all $\mathbf{a}_j \in \mathbf{B}$:

Further, it can be verified that $z_j - c_j = 0$ for all j for which $\mathbf{a}_j$ is in $\mathbf{B}$. that is, for $j = 1, 4$. Since $\mathbf{Bx}_1 = \mathbf{a}_1$, that is,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_{11} \\ x_{11} + x_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or $$x_{11} = 1, x_{21} = 0.$$

Therefore, $$\mathbf{x}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = (1, 0) = \mathbf{e}_1.$$

Similarly, $$\mathbf{x}_4 = \mathbf{B}^{-1}\mathbf{a}_4 = (0, 1) = \mathbf{e}_2$$

Hence $$z_1 - c_1 = \mathbf{C_B x}_1 - c_1 = (1, 0)(1, 0) - 1 = 0$$
$$z_4 - c_4 = \mathbf{C_B x}_4 - c_4 = (1, 0)(0, 1) - 0 = 0.$$

From this it is verified that $z_j - c_j$ is always zero for those j for which $\mathbf{a}_j$ is in $\mathbf{B}$. Hence, there is no need to compute such $z_j - c_j$ because these values will always come out to be zero.

5·7 ALTERNATIVE OPTIMAL SOLUTIONS

Definition. *If the set of variables giving the optimal value of the objective function is not unique, alternative optimum solutions to the given linear programming problem exist.*

In the graphical solution of linear programming problem (*chapter* 4, *Example* 14) infinite number of points from A to B (Fig. 4·5) give the optimum solutions.

Further, this case has already been generalized in the second part of the *Theorem 5·14* which, in other words, states that *if there are two or more distinct optimal basic feasible solutions, there will be an infinite number of optimal solutions to such linear programming problems. However, among these infinite optimal solutions some may be non-basic in the first case and some may be basic in the second case of the following theorem.*

Theorem 5·11. (Conditions for Alternative Optimum Solutions)

(i) *If there is an optimal basic feasible solution to a linear programming problem and, for some $\mathbf{a}_j$ not in $\mathbf{B}$, $z_j - c_j = 0$, $x_{ij} < 0$ for all $i = 1, 2, ..., m$, then non-basic alternative optimum solution will exist.*

(ii) *Secondly, if $z - c_j = 0$ for some $\mathbf{a}_j$ not in $\mathbf{B}$ and $x_{ij} > 0$ for at least one i, then an alternative basic optimum solution will exist.*

Proof. ***First Part.***

Following the proof of *Theorem 5·9* down to result (5·48),

$$\hat{z} = z^* - \lambda (z_j - c_j)$$

$$\left(\text{since } z^* = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} \text{ for given optimal solution}\right)$$

Since $z_j - c_j = 0$ for some $\mathbf{a}_j$ not in $\mathbf{B}$, therefore $\hat{z} = z^*$.

This shows that the non-basic feasible solution given by

$$\hat{x}_i = x_{Bi} - \lambda\, x_{ij},\ i = 1, 2, \ldots, m$$
$$\hat{x}_{m+1} = \lambda$$

when $x_{ij} < 0$ $(i = 1, 2, \ldots, m)$ and $\lambda > 0$, gives the same value of z. Hence this new solution with $m + 1$ number of positive variables gives alternative optimum non-basic feasible solutions for arbitrary value of positive scalar λ.

Second Part. Since in this case $x_{ij} > 0$ for at least one i, one β_r can be replaced by $\mathbf{a}_j$ not in B and obtain another basic feasible solution (as explained in *Section 3·5*) which gives the value of the objective function as

$$\hat{z} = z^* + \frac{x_{Br}}{x_{rj}}(z_j - c_j) \qquad \text{[from equation (5·36)]}$$

$$= z^* \text{ (because } z_j - c_j = 0)$$

Since the value of the objective function remains unaltered, an alternative optimum basic feasible solutions are obtained.

Remarks :

1. In case, $\mathbf{X_B}$ is degenerate solution and $z_j - c_j > 0$ for $\mathbf{a}_j$ not in **B**, $x_{Br} = 0$ and $x_{rj} > 0$, then an alternative optimum basic feasible solution exists which can be proved as above after dealing with the problem of degeneracy.
2. Further, notice that if $\mathbf{X_B}$ is non-degenerate and $z_j - c_j > 0$ for all $\mathbf{a}_j$ not in **B**, then only unique optimal solution exists. It is necessary that this unique optimal solution should be basic.

5·8 CHANGE OF OBJECTIVE FUNCTION FROM MINIMIZATION TO MAXIMIZATION

In order to formulate the LP problem in standard form of simplex method, the objective function of minimization can be easily converted to maximization one by using the following *Minimax Theorem.*

Theorem 5·12. (Minimax Theorem). *Let f be a linear function of n variables such that $f(\mathbf{X}^*)$ is its minimum value for some point $\mathbf{X}^*, \mathbf{X}^* \in R^n$. Then, $-f(\mathbf{X})$ attains its maximum at the point $\mathbf{X}^*$. Moreover, for $\mathbf{X} \in R^n$*

$$\text{Min} f(\mathbf{X}) = -\,Max\,\{-f(\mathbf{X})\}.$$

Proof. Since $f(\mathbf{X})$, is a minimum at the point $\mathbf{X}^*$,

$$f(\mathbf{x}^*) \le f(\mathbf{x}), \text{ for all } \mathbf{x} \in R^n$$

or

$$-f(\mathbf{X}^*) \ge -f(\mathbf{X}), \text{ for all } \mathbf{X} \in R^n.$$

Q. 1. Given and L.P.P. explain under what conditions

(i) a B.F.S. can be improved (ii) a B.F.S. is optimal, and (iii) the solution is unbounded.

2. Can a vector that is inserted at one iteration in simplex method be removed immediately at the next iteration ? When can this occur and when is it impossible ?
3. While solving any L.P.P. by simplex method suppose at some step you have a degenerate basic feasible solution. Under what conditions, at the next step, a non-degenate BFS can be obtained ?
4. While solving a linear programming problem by simplex method, what indicates each of the following situations :

   (i) Unbounded solution

   (ii) Alternate optimal basic feasible solution

   (iii) Presence of redunant constraints ?
5. Show that any vector which is removed from the basis at one iteration in the simplex method for solving a linear programming problem cannot re-enter at the next iteration.
6. Show that a sufficient condition for a basic feasible solution to an L.P.P. to be a maximum solution is that $z_j - c_j > 0$ for all j for which the column vector $a_j \in A$ is not in the basis **B**.
7. If there is an optimal basic feasible solution to a linear programming problem and, for some a_j not in the basis, $z_j - c_j = 0$, $y_{ij} \le 0$ for all i, then non-basic alternative optima will exist.
8. Given a basic feasible solution $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ to the linear programming problem $\mathbf{AB} = \mathbf{b}$, $\mathbf{X} \ge 0$. Maximize $z = \mathbf{CX}$, such that $z_j - c_j \ge 0$ for all columns a_j in **A**. Show that the basic feasible solution is an optimal basic feasible solution.
9. Let $\mathbf{X_B}$ be a basic feasible solution obtained by admitting a non-basic column vector $\mathbf{a}_j$ in the basis, for which the net evaluations $z_j - c_j$ is negative. Then show that $\mathbf{X_B}$ is an improved basic feasible solution to the problem.
10. If for any basic feasible solution to an L.P.P., there is some column $\mathbf{a}_j$ not in the basis for which

    $z_j - c_j < 0$ and $y_{ij} \le 0$ $(i = 1, 2, \ldots, m)$

    then prove that there exists a feasible solution in which $m + 1$ variables can be different from zero with the value of the objective function arbitrarily large. Also, show that in each case, the problem has an unbounded solution if the objective function is to be maximized.
11. Let there exist a basic feasible solution to a l.p.p. If for at least one j, for which $y_{ij} \le 0$ $(i = 1, 2, \ldots, m)$, $z_j - c_j < 0$, then prove that there exist no optimum solution to this l.p.p.

Applications of Simplex Method

5·9 BASIC CONCEPTS

Here the theoretical results obtained in earlier be used for computational development of simplex method. In this method, we make from initial basic feasible solution (extreme point) to new one (having a value of z, at least as large as the preceding one) until an optimal solution is reached.

The details of simplex method are developed for solving standard LPP : Max $z = ax$, subject to $Ax = b$, $x \geq 0$ wjere A is max mateix. For convenience, we shhall take maximization problem only.

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such cases, a simple and most widely used simplex method is adopted which was developed by ***G. Dantzig*** in 1947.

The *simplex method*† provides an ***algorithm*** (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the ***fundamental theorem of linear programming.***

It is clear from Fig. 4.4 (page 76) that feasible solutions may be *infinite* in number (because there are infinite number of points in the feasible region, *OABCD*). So, it is rather impossible to search for the optimum solution amongst all the feasible solutions. But fortunately, the number of basic feasible solutions are finite in number (which are corresponding to extreme points O, A, B, C, D, respectively). Even then, a great labour is required in finding all the basic feasible solutions and to select that one which optimizes the objective function.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The procedure is explained in detail through a numerical example (*see Example* 2, page 116).

The simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of—

(i) having a trial basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

The computational procedure requires at most m [equal to the number of equations in (4·12)] non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations the degeneracy arises in LP problem. The case of degeneracy has also been discussed in detail in *this chapter*.

Further, it is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way. One of the non-basic variables (which are zero now) at one iteration becomes *basic* (non-zero) at the following iteration, and is called an ***entering variable.*** To compensate, one of the basic variables (which are non-zero now) at one iteration becomes non-basic (zero) at the following iteration, and is called a ***departing variable.*** The other non-basic variables remain zero, and the other basic variables, in general, remain non-zero (though their values may change).

For convenience, re-state the LP problem in standard form :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m} \quad \ldots(5.52)$$

Subject to the constraints :

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & + x_{n+1} & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & & + x_{n+2} & = b_2 \\ \ldots \quad \ldots \quad \ldots & \ldots & \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \quad \ldots(5.53)$$

and
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0 \quad \ldots(5.54)$$

For easiness, an obvious starting basic feasible solution of m equations (5·53) is usually taken as : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; $x_{n+1} = b_1, x_{n+2} = b_2, \ldots, x_{n+m} = b_m$. For this solution, the value of the objective function (5·52) is zero. Here $x_1, x_2, x_3, \ldots, x_n$ (each equal to zero) are ***non-basic variables*** and remaining variables $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m})$ are ***basic variables*** (some of them may also have the value zero).

5.10 SOME MORE DEFINITIONS AND NOTATIONS

The first basic feasible solution is : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; and $x_{n+1} = b_1$, $x_{n+2} = b_2$, $x_{n+3} = b_3 \ldots$, $x_{n+m} = b_m$ for the reformulated LP problem : Max $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$ and $\mathbf{X} \geq \mathbf{0}$.

First denote the jth column of $m \times (n+m)$ matrix **A** by $\mathbf{a}_j$ ($j = 1, 2, 3, \ldots, n+m$), so that

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m+n}] . \qquad \ldots(5.54)$$

Now form an $m \times m$ non-singular matrix **B**, called ***basis matrix***, whose column vectors are m linearly independent columns selected from matrix **A** and renamed as $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$. Therefore,

$$\mathbf{B} = [\beta_1, \beta_2, \ldots, \beta_m] = [\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \ldots, \mathbf{a}_{n+m}] \qquad \ldots(5.55)$$

For initial basic feasible solution,

$$\mathbf{B} = [(1, 0, 0, \ldots, 0), (0, 1, 0, 0 \ldots, 0), \ldots, (0, 0, \ldots, 1)] = I_m \text{ (identity matrix)}.$$

The matrix **B** is evidently a basis matrix because column vectors in **B** form a basis set of m-dimensional Euclidean space (E^m).

Second, denote the basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ by $x_{B1}, x_{B2}, \ldots, x_{Bm}$ respectively, to give the basic feasible solution in the form :

$$\mathbf{X}_B = (x_{B1}, x_{B2}, x_{B3}, \ldots, x_{Bm}) = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m}) \qquad \ldots(5.56)$$

For initial basic feasible solution,

$$\mathbf{X}_B = (b_1, b_2, b_3, \ldots, b_m) = \text{right side constants of (5.53)}.$$

Next, the coefficients of basic variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ in the objective function z will be denoted by $c_{B1}, c_{B2}, \ldots, c_{Bm}$ respectively, so that

$$\mathbf{C}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}).$$

For initial basic feasible solution,

$$\mathbf{C}_B = (0, 0, \ldots, 0) = \mathbf{0} \text{ (null vector)}$$

Consequently, the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m} \text{ becomes}$$

$$z = c_1.0 + c_2.0 + \ldots + c_n.0 + c_{B1}x_{B1} + \ldots + c_{Bm}x_{Bm} \qquad [\text{since } x_1 = x_2 = x_3 = \ldots = x_n = 0]$$

or

$$z = \mathbf{C}_B \mathbf{X}_B . \qquad \ldots(5.57)$$

Because $\mathbf{C}_B = \mathbf{0}$ (null vector) for initial solution, therefore

$$z = 0, \mathbf{X}_B = \mathbf{b} .$$

Since **B** is an $m \times m$ non-singular basis matrix, any vector in E^m can be expressed as a linear combination of vectors in **B** (by definition of basis for vector space). In particular, each vector $\mathbf{a}_j$ ($j = 1, 2, \ldots, n+m$) of matrix **A** can be expressed as a linear combination of vectors β_i ($i = 1, 2, \ldots, m$) in **B**. The notation for such linear combination is given by

$$\mathbf{a}_j = x_{1j}\beta_1 + x_{2j}\beta_2 + \ldots + x_{mj}\beta_m = (\beta_1, \beta_2, \ldots, \beta_m)\begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} = \mathbf{B}\mathbf{X}_j \qquad \ldots(5.58)$$

where x_{ij} ($i = 1, 2, 3, \ldots, m$) are scalars required to express each $\mathbf{a}_j$ ($j = 1, 2, 3, \ldots, n+m$) as linear combination of basis vectors $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$.

Therefore, $\mathbf{X}_j = \mathbf{B}^{-1}\mathbf{a}_j$ and hence matrix ($\mathbf{X}_j$) will change if the columns of (**A**) forming (**B**) change.

For initial solution, $\mathbf{a}_j = \mathbf{I}_m\mathbf{X}_j = \mathbf{X}_j$.

Next define a new variable, say z_j , as

$$z_j = x_{1j}c_{B1} + x_{2j}c_{B2} + \ldots + x_{mj}c_{Bm} = \sum_{i=1}^{m} c_{Bi}x_{ij} = \mathbf{C}_B\mathbf{X}_j . \qquad \ldots(5.59)$$

Δ_j denotes the *net evaluation* which is computed by the formula :

$$\Delta_j = z_j - c_j = \mathbf{C}_B\mathbf{X}_j - c_j \qquad \ldots(5.60)$$

Lastly, these notations can be summerized in the following Starting Simplex Table 5.51.

Table 5.1 : Starting Simplex Table

		$c_j \rightarrow$	c_1	c_2	...	c_n	0	0	...	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1 (= \mathbf{a}_1)$	$\mathbf{X}_2 (= \mathbf{a}_2)$	...	$\mathbf{X}_n (= \mathbf{a}_n)$	$\mathbf{X}_{n+1}$ (β_1)	$\mathbf{X}_{n+2}$ (β_2)	...	$\mathbf{X}_{n+m}$ (β_m)	MIN RATIO
$x_{n+1} (= s_1)$	$c_{B1} (= 0)$	$x_{B1} (= b_1)$	$x_{11} (= a_{11})$	$x_{12} (= a_{12})$	...	$x_{1n} (= a_{1n})$	1	0	...	0	
$x_{n+2} (= s_2)$	$c_{B2} (= 0)$	$x_{B2} (= b_2)$	$x_{21} (= a_{21})$	$x_{22} (= a_{22})$	...	$x_{2n} (= a_{2n})$	0	1	...	0	
:	:	:	:	:	:	:	:	:	:	:	
$x_{n+m} (= s_m)$	$c_{Bm} (= 0)$	$x_{Bm} (= b_m)$	$x_{m1} (= a_{m1})$	$x_{m2} (= a_{m2})$	...	$x_{mn} (= a_{mn})$	0	0	...	1	
	$z = \mathbf{C_B X_B}$		Δ_1	Δ_2	...	Δ_n	0	0	...	0	$\Delta_j = \mathbf{C_B X}_j - c_j$

Note. Basic variables in the first column are always sequenced in the order of columns forming the unit matrix.

Above definitions and notations can be clearly understood by the following numerical example.

5.10-1 An Example to Explain Above Definitions and Notations

Example 1. *Illustrate definitions and notations by the linear programming problem :*

Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$, *subject to* $4x_1 + 2x_2 + x_3 + x_4 = 4,\ x_1 + 2x_2 + 3x_3 - x_5 = 8$.

Solution. First of all, constraint equations in matrix form may be written as

$$\overset{\mathbf{A}}{\overset{\begin{matrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5\end{matrix}}{\begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix}}} \overset{\mathbf{X}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \overset{\mathbf{B}}{\begin{bmatrix} 4 \\ 8 \end{bmatrix}}$$

or $$\mathbf{AX} = \mathbf{b}.$$

A basis matrix $\mathbf{B} = (\beta_1, \beta_2)$ is formed using columns $\mathbf{a}_3$ and $\mathbf{a}_1$, so that

$$\beta_1 = \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ \beta_2 = \mathbf{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rank of matrix A is 2, and hence $\mathbf{a}_3$, $\mathbf{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus, basis matrix is $$\mathbf{B} = (\beta_1, \beta_2) = \overset{\begin{matrix}\mathbf{a}_3 & \mathbf{a}_1\end{matrix}}{\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}}$$

Using (5·4) and (5.8), the basic feasible solution is

$$\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b} = \left[\frac{1}{|\mathbf{B}|} \text{adj}\ (\mathbf{B})\right]\mathbf{b} = \frac{-1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or $$\mathbf{X_B} = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}.$$

Therefore, basic variables are $x_{B1} = 28/11 = x_3$, $x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero) *i.e.*, $x_2 = x_4 = x_5 = 0$. Also,

$$c_{B1} = \text{coefficient of } x_{B1} = \text{coeff. of } x_3 = c_3 = 3$$

$$c_{B2} = \text{coefficient of } x_{B2} = \text{coeff. of } x_1 = c_1 = 1$$

Hence $$\mathbf{C_B} = (3, 1).$$

Now, using (5.7), the value of the objective function is

$$z = \mathbf{C_B X_B} = (3,\ 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a_j} = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_1 + x_{22}\beta_2 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5.3) to get

$$\mathbf{X}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $\quad x_{12} = 6/11,\ x_{22} = 4/11.$

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5.6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C_B X_2} = (3,\ 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly z_1, z_3, z_4, z_5 can also be computed.

5·11 COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of the simplex procedure is first explained by the following simple example.

Example 2. *Consider the linear programming problem :*

Maximize $z = 3x_1 + 2x_2$, subject to the constraints :

$$x_1 + x_2 \le 4,\ x_1 - x_2 \le 2,\ and\ x_1, x_2 \ge 0.$$

[Kanpur 2000, 96; IAS (Maths.) 92]

Solution. ***Step 1.*** First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the b_i's (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '$\le$', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

Maximize $\quad z = 3x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints :

$$x_1 + x_2 + s_1 \quad = 4$$
$$x_1 - x_2 \quad + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained in ***Sec 5·2.***

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from *Table* 5·2 as : $s_1 = 4,\ s_2 = 2,\ x_1 = 0,\ x_2 = 0$, and the value of the objective function is *zero.*

Note. In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called *basic variables*. Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.

Table 5·2 : Starting Simplex Table

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	$X_3 (S_1)$ (β_1)	$X_4 (S_2)$ (β_2)	MIN. RATIO X_B/X_k for $X_K > 0$
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN NEXT STEP.
s_2	0	2	1	−1	0	1	
	$z = C_B X_B$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - c_j = C_B X_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector X_j) by the formula

$$\Delta_j = z_j - c_j = C_B X_j - c_j \qquad \text{[from (5·10)]}$$

Thus, we get

$\Delta_1 = C_B X_1 - c_1$	$\Delta_2 = C_B X_2 - c_2$	$\Delta_3 = C_B X_3 - c_3$	$\Delta_4 = 0$
$= (0, 0)(1, 1) - 3$	$= (0, 0)(1, -1) - 2$	$= (0, 0)(1, 0) - 0$	
$= (0 \times 1 + 0 \times 1) - 3$	$= (0 \times 1 - 0 \times 1) - 2$	$= (0 \times 1 + 0 \times 0) - 0$	
$= -3$	$= -2$	$= 0$	

Remark. Note that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test :

(i) If all $\Delta_j\ (= z_j - c_j) \geq 0$, the solution under test will be *optimal. Alternative optimal solutions* will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero (≤ 0), then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in ***Step 6.***

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector X_k is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min\ [\Delta_1, \Delta_2] = \min\ [-3, -2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector X_1 must enter the basis matrix. The column X_1 is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of X_B by the corresponding *positive* elements of predetermined incoming vector X_k. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Br}}{x_{ik}},\ x_{ik} > 0\right]$$

For $k = 1$,

$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}, \frac{2}{1}\right]$$

or

$$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, X_4 marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Table*** **5·2** is now modified to ***Table*** **5·3** given below.

Table 5·3

		$c_j \to$	3	2	0 (key element)	0	
BASIC VARIABLES	Ca	Xa	X_1	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$ (β_2)	MIN. RATIO (X_B/X_1)
s_1	0	4	1	1	1	0	4/1
s_2	0	2	[1]	1	0	1	2/1 ← MIN. RATIO
		$z = C_B X_B = 0$	−3 (min. Δ_j)	−2	0	0	$\leftarrow \Delta_j = z_j - c_j = C_B B_j - c_j$
			↑ entering vector			↓ leaving vector	

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '□' position and zero at all other places of X_1. If the number in the marked '□' position is other than *unity, divide all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow (←) and incoming vector arrow (↑) is called the **key element or pivot element**).*

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column X_1. Thus, the process can be fortified by simple matrix transformation as follows:

The intermediate coefficient matrix is:

	X_B	X_1	X_2	X_3	X_4	
R_1	4	1	1	1	0	
R_2	2	1	−1	0	1	
R_3	z = 0	−3	−2	0	0	$\leftarrow \Delta_j$

Apply $R_1 \to R_1 - R_2$, $R_3 \to R_3 + 3R_2$ to obtain

X_B	X_1	X_2	X_3	X_4	
2	0	2	1	−1	
2	1	−1	0	1	
z = 6	0	−5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table as follows:

Table 5·4

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_2)	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$	MIN-RATIO $(X_B/X_2, X_2 > 0)$
s_1	0	2	0	[2]	1	−1	$\frac{2}{2}$ ← key row
x_1	3	2	1	−1	0	1	~~$\frac{2}{-1}$~~ (negative ratio is not counted)
		$z = C_B X_B = 6$	0	−5 ↑	0 ↓	3	$\leftarrow \Delta_j$
				key column			

From this table, the improved basic feasible solution is read as : $x_1 = 2$, $x_2 = 0$, $s_1 = 2$, $s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_B X_j - c_j$.

Step 8. Now repeat *Steps* 5 *through* 7 as and when needed until an optimum solution is obtained in Table 5·5.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence $\mathbf{X}_2$ should be the entering vector (key column). By minimum ratio rule :

$$\text{Minimum Ratio}\left(\frac{\mathbf{X_B}}{\mathbf{X}_2}, \mathbf{X}_2 > 0\right) = \text{Min}\left[\frac{2}{2}, -\right]$$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	X_B	X_1	X_2	X_3	X_4	
R_1	1	0	[1]	1/2	− 1/2	
R_2	2	1	− 1	0	1	
R_3	z = 6	0	− 5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 5R_1$

1	0	1	1/2	− 1/2	
3	1	0	1/2	1/2	
z = 11	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follows :

Final Simplex Table 5·5

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_2)	X_2 (β_1)	S_1	S_2	
$\to x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = C_B X_B = 11$		0	0	5/2	1/2	$\leftarrow \Delta_j$

The solution as read from this table is : $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3 , x_1 = 1 , \max z = 11 .$$

Note. If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 3·4, i.e., $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j.

5·12 SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with its different computational steps can be more conveniently represented by the following single table (see Table 5.6).

Table 5·6

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1	S_2	MIN RATIO (X_B/X_K)
s_1	0	4	1	1	1	0	4/1
$\leftarrow s_2$	0	2	[1]	−1	0	1	2/1 ← Min
$x_1 = x_2 = 0$	$z = C_B X_B = 0$		−3* ↑	−2	0	0 ↓	$\leftarrow \Delta_j = z_j - c_j$
$\leftarrow s_1$	0	2	0	[2]	1	−1	2/2 Min ←
$\to x_1$	3	2	1	−1	0	1	—
$x_2 = s_2 = 0$	$z = C_B X_B = 6$		0	−5* ↑	0 ↓	3	$\leftarrow \Delta_j$
$\to x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = C_B X_B = 11$		0	0	5/2	1/2	← All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

Q. 1. What is a simplex ? Describe simplex method of solving linear programming problems.
2. Write the steps used in the simplex method. [Kanpur (B. Sc.) 90]
3. Describe a computational procedure of the simplex method for the solution of a maximization l.p.p.

Tips for Quick Solution :

1. In the first iteration only, since Δ_j's are the same as $-c_j$'s, so there is no need of calculating them separately by using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$.
2. Mark *min* (Δ_j) by '↑' which at once indicates the column $\mathbf{X}_k$ needed for computing the minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$.
3. '*Key element*' is found at the place where the upward directed arrow '↑' of min Δ_j and the left directed arrow (←) of minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$ intersect each other in the simplex table.
4. '*Key element*' indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become 0.
5. Since Δ_j's corresponding to unit column vectors are always zero, there is no need of calculating them.
6. While transforming the table by row operations, the value of z and corresponding Δ_j's are also computed at the same time. Thus a lot of time and labour can be saved in adopting this technique.

Example 3. *Min* $z = x_1 - 3x_2 + 2x_3$, *subject to* :

$$3x_1 - x_2 + 3x_3 \le 7, -2x_1 + 4x_2 \le 12, -4x_1 + 3x_2 + 8x_3 \le 10, \text{ and } x_1, x_2, x_3 \ge 0.$$

[Kanpur (B.Sc.) 95, 93, (B.A.) 90]

Solution. This is the problem of minimization. Converting the objective function from minimization to maximization, we have

$$\text{Max.} -z = -x_1 + 3x_2 - 2x_3 = \text{Max. } z' \text{ where } -z = z',$$

Here we give only tables of solution. The students are advised to verify them.

Table 5·7. Simplex Table

		$c_j \rightarrow$	−1	3	−2	0	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X}_k)$
x_4	0	7	3	−1	3	1	0	0	—
$\leftarrow x_5$	0	12	−2	[4]	0	0	1	0	12/4 ← min.
x_6	0	10	−4	3	8	0	0	1	10/3
$x_1 = x_2 = x_3 = 0$	$z' = 0, z = 0$		1	−3* ↑	2	0 ↓	0	0	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	10	[5/2]	0	3	1	1/4	0	$\frac{10}{5/2}$ ←
$\rightarrow x_2$	3	3	−1/2	1	0	0	1/4	0	—
x_6	0	1	−5/2	0	8	0	−3/4	1	—
$x_1 = x_3 = x_5 = 0$	$z' = 9$ ∴ $z = -9$		−1/2* ↑	0	2	0 ↓	3/4	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	−1	4	1	0	6/5	2/5	1/10	0	
x_2	3	5	0	1	3/5	1/5	3/10	0	
x_6	0	11	0	0	11	1	−1/2	1	
$x_3 = x_4 = x_2 = 0$	$z' = 11$ ∴ $z = -11$		0	0	13/5	1/5	8/10	0	$\leftarrow \Delta_j \ge 0$

The optimal solution is : $x_1 = 4, x_2 = 5, x_3 = 0$, *Min* $z = -11$.

Example 4. *Max.* $z = 3x_1 + 2x_2 + 5x_3$, *subject to the constraints :*

$$x_1 + 2x_2 + x_3 \le 430, 3x_1 + 2x_3 \le 460, x_1 + 4x_2 \le 420, \text{ and } x_1, x_2, x_3 \ge 0.$$

[IAS (Main 94)]

Solution.

Table 5·8. Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (3)	X_2 (2)	X_3 (5)	X_4 (0)	X_5 (0)	X_6 (0)	MIN RATIO (X_B/X_k)
x_4	0	430	1	2	1	1	0	0	430/1
$\leftarrow x_5$	0	460	3	0	[2]	0	1	0	460/2 ←
x_6	0	420	1	4	0	0	0	1	—
$x_1=x_2=x_3=0$	$z=0$		−3	−2	−5* ↑	0	0 ↓	0	$\leftarrow\Delta_j$
$\leftarrow x_4$	0	200	−1/2	[2]	0	1	−1/2	0	200/2 ←
$\rightarrow x_3$	5	230	3/2	0	1	0	1/2	0	—
x_6	0	420	1	4	0	0	0	1	420/4
$x_1=x_2=x_5=0$	$z=1150$		9/2	−2* ↑	0	0 ↓	5/2	0	$\leftarrow\Delta_j$
$\rightarrow x_2$	2	100	−1/4	1	0	1/2	−1/4	0	
x_3	5	230	3/2	0	1	0	1/2	0	
x_6	0	20	2	0	0	−2	1	1	
$x_1=x_4=x_5=0$	$z=1350$		4	0	0	1	2	0	$\leftarrow\Delta_j\geq 0$

Since all $\Delta_j \geq 0$, the solution is : $x_1=0$, $x_2=100$, $x_3=230$, max $z=1350$.

Example 5. *Solve the LP problem :Max.* $z=3x_1+5x_2+4x_3$, *subject to the constraints :*

$$2x_1+3x_2\leq 8,\ 2x_2+5x_3\leq 10,\ 3x_1+2x_2+4x_3\leq 15,\ \textit{and}\ x_1, x_2, x_3\geq 0.$$

[Tamilnadu (ERODE) 97; Rewa 93; Kanpur (B.Sc.) 92; (B.A.) 90, Meerut (M.Sc. Stat. & B.Sc. Math.) 90]

Solution. After introducing slack variables, the constraint equations become :

$$\begin{aligned} 2x_1+3x_2 \quad +x_4 &= 8\\ 2x_2+5x_3 \quad +x_5 &= 10\\ 3x_1+2x_2+4x_3 \quad +x_6 &= 15. \end{aligned}$$

Table 5·9. Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (3)	X_2 (5)	X_3 (4)	X_4 (0)	X_5 (0)	X_6 (0)	MIN RATIO. (X_B/X_2)
$\leftarrow x_4$	0	8	2	[3]	0	1	0	0	8/3 ←
x_5	0	10	0	2	5	0	1	0	10/2
x_6	0	15	3	2	4	0	0	1	15/2
$x_1=x_2=x_3=0$	$z=C_BX_B=0$		−3	−5* ↑ Incoming vector	−4	0 ↓ outgoing vector	0	0	$\leftarrow\Delta_j$

Now apply short-cut method for minimum ratio rule (min X_B/X_2), and find the key element 3. This key element indicates that unity should be at first place of X_2, so the vector to be removed from the basis matrix is X_4.

Now, in order to get the second simplex table, calculate the intermediate coefficient matrices as follows :

First, divide the first row by 3 to get

R_1	8/3	2/3	1	0	1/3	0	0	
R_2	10	0	2	[5]	0	1	0	
R_3	15	3	2	4	0	0	1	
R_4	0	−3	−5	−4	0	0	0	$\leftarrow\Delta_j$

Applying $R_2\rightarrow R_2-2R_1$, $R_3\rightarrow R_3-2R_1$, $R_4\rightarrow R_4+5R_1$,

R_1	8/3	2/3	[1]	0	1/3	0	0	
R_2	14/3	– 4/3	0	5	– 2/3	1	0	
R_3	29/3	5/3	0	4	– 2/3	0	1	
R_4	40/3	1/3	0	-4	5/3	0	0	$\leftarrow \Delta_j$

Now the second simplex table (Table 5.10) is constructed as below :

Table 5·10

		$c_j \rightarrow$	3	5	4	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO (X_B/X_3)
$\rightarrow x_2$	5	8/3	2/3	1	0	1/3	0	0	—
$\leftarrow x_5$	0	14/3	– 4/3	0	[5]	–2/3	1	0	$\frac{14}{3}/5 \leftarrow$
x_6	0	29/3	5/3	0	4	– 2/3	0	1	$\frac{29}{3}/4$
$x_4 = x_1 = x_3 = 0$		$z = \frac{40}{3}$	1/3	0	–4* ↑	5/3	0 ↓	0	$\leftarrow \Delta_j$

Incoming Outgoing

Now verify that

$$\Delta_1 = C_B X_1 - c_1 = -3 + (5, 0, 0)(2/3, -4/3, 5/3) = 1/3$$

$$\Delta_3 = C_B X_3 - c_3 = -4 + (5, 0, 0)(0, 5, 4) = -4$$

$$\Delta_4 = C_B X_4 - c_4 = 0 + (5, 0, 0)(1/3, -2/3, -2/3) = 5/3.$$

The key-element is found to be 5. Hence the vector to be removed from the basis matrix is X_5. Thus proceeding exactly in the same manner, the remaining simplex tables are obtained (Tables 5.11 and 5.12).

Table 5·11

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{2}{3}/\frac{2}{3}$
$\rightarrow x_3$	4	14/15	– 4/15	0	1	– 2/15	1/15	0	—
$\leftarrow x_6$	0	89/15	[41/15]	0	0	– 2/15	– 4/5	1	$\frac{89}{15}/\frac{41}{15} \leftarrow$
$x_1 = x_5 = x_4 = 0$		$z = 256/15$	– 11/15* ↑	0	0	17/15	4/5	0 ↓	$\leftarrow \Delta_j$

Incoming Outgoing

Table 5·12. Final Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	50/41	0	1	0	15/41	8/41	– 10/41	
x_3	4	62/41	0	0	1	– 6/41	5/41	4/41	
x_1	3	89/41	1	0	0	– 2/41	– 12/41	15/41	
$x_4 = x_5 = x_6 = 0$		$z = C_B X_B = 765/41$	0	0	0	45/41	24/41	11/41	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution given by $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$, max $z = 765/41$, is optimal.

Example 6. *Minimize* $z = x_2 - 3x_3 + 2x_5$, *subject to the constraints :*

$$3x_2 - x_3 + 2x_5 \leq 7, -2x_2 + 4x_3 \leq 12, -4x_2 + 3x_3 + 8x_5 \leq 10, \textit{ and } x_2, x_3, x_5 \geq 0.$$

[JNTU (Mech.) 99; Kanpur 96; Madurai B.Sc. (Comp. Sc.) 92, (Appl. Math) 85; Kerala B.Sc. (Math.) 90]

Solution. Equivalently, max $z' = -x_2 + 3x_3 - 2x_5$ where $z' = -z$. Introducing x_1, x_4 and x_6 as slack variables, the constraint equations become :

$$x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 = 7$$

$$0x_1 - 2x_2 + 4x_3 + x_4 + 0x_5 + 0x_6 = 12$$

$$0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 = 10.$$

Now proceeding as in above example the simplex computations are performed as follows :

Table 5·13

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO (X_B/X_k)
		$c_j \rightarrow$	0	−1	3	0	−2	0	
x_1	0	7	1	3	−1	0	2	0	—
$\leftarrow x_4$	0	12	0	−2	[4]	1	0	0	12/4 ←
x_6	0	10	0	−4	3	0	8	1	10/3
		$z' = 0$	0	1	−3* ↑	0 ↓	2	0	$\leftarrow \Delta_j$
$\leftarrow x_1$	0	10	1	[5/2]	0	−1/4	2	0	4 ←
$\rightarrow x_3$	3	3	0	−1/2	1	1/4	0	0	—
x_6	0	1	0	−5/2	0	−3/4	8	1	—
		$z' = 9$	0 ↓	−1/2* ↑	0	3/4	2	0	$\leftarrow \Delta_j$
x_2	−1	4	2/5	1	0	1/10	4/5	0	
x_3	3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	−1/2	10	1	
		$z' = 11$ or $z = -11$	1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_j \geq 0$

Thus, optimal solution is : $x_2 = 4$, $x_3 = 5$, $x_5 = 0$, min. $z = -11$.

Alternative forms of Example 6 :

(i) *Min.* $z = x_1 - 3x_2 + 2x_3$, *subject to* $3x_1 - x_2 + 2x_3 \leq 7$, $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$ *and* $x_1, x_2, x_3 \geq 0$.

(ii) *Min.* $z = x_2 - 3x_3 + 2x_5$, *subject to the constraints :*

$$x_1 + 3x_2 - x_3 + 2x_5 = 7, -2x_2 + 4x_3 + x_4 = 12, -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \text{ and } x_1, x_2, \ldots, x_6 \geq 0.$$

Example 7 (Bounded Variables). *A manufacturer of three products tries to follow a policy of producing those which continue most to fixed cost and profit. However, there is also a policy of recognising certain minimum sales requirements currently, these are :*

Product :	X_1	X_2	X_3
Units per week :	20	30	60

There are three producing departments. The product times in hour per unit in each department and the total times available for each week in each department are :

Departments	*Time required per product in hours*			*Total hours available*
	X_1	X_2	X_3	
1	0·25	0·20	0·15	420
2	0·30	0·40	0·50	1048
3	0·25	0·30	0·25	529

The contribution per unit of product X_1, X_2, X_3 *is Rs.* 10·50, *Rs.* 9·00 *and Rs.* 8·00 *respectively. The company has scheduled 20 units of* X_1, *30 units of* X_2 *and 60 units of* X_3 *for production in the following week, you are required to state :*

(i) *Whether the present schedule is an optimum one from a profit point of view and if it is not, what it should be;*

(ii) *The recommendations that should be made to the firm about their production facilities (following the answer to (i) above).*

Solution. The formulation of the problem is as follows :

Maximize $z = 10{\cdot}5X_1 + 9X_2 + 8X_3$, *subject to the constraints :*

$$0{\cdot}25X_1 + 0{\cdot}20X_2 + 0{\cdot}15X_3 \leq 420$$

$$0{\cdot}30X_1 + 0{\cdot}40X_2 + 0{\cdot}50X_3 \leq 1048$$

$$0{\cdot}25X_1 + 0{\cdot}30X_2 + 0{\cdot}25X_3 \le 529$$
$$0 \le X_1 \ge 20\,,\ 0 \le X_2 \ge 30\,,\ 0 \le X_3 \ge 60\,.$$

Since the company is already producting minimum of X_2 and X_3 it should, at least, produce maximum of X_1 limited by the first constraint. Lower bounds are specified in this problem, i.e., $X_1 \ge 20$, $X_2 \ge 30$, $X_3 \ge 60$. This can be handled quite easily by introducing the new variables x_1, x_2 and x_3 such that

$$X_1 = 20 + x_1, X_2 = 30 + x_2, X_3 = 60 + x_3\,.$$

Substituting for X_1, X_2 and X_3 in terms of x_1, x_2, x_3, the problem now becomes:

Maximize $z = 10{\cdot}5x_1 + 9x_2 + 8x_3 +$ constant, *subject to the constraints*: $0{\cdot}25x_1 + 0{\cdot}20x_2 + 0{\cdot}15x_3 \le 400$, $0{\cdot}30x_1 + 0{\cdot}40x_2 + 0{\cdot}50x_3 \le 1000$, $0{\cdot}25x_1 + 0{\cdot}30x_2 + 0{\cdot}25x_3 \le 500$, *and* $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

The students may now proceed to find the optimal solution by simplex method in the usual manner.

Example 8. *For a company engaged in the manufacture of three products, viz.* X, Y *and* Z, *the available data are given below:*

Minimum Sales Requirement

Product:	X	Y	Z
Min. sales requirement per month:	10	20	30

Operations, Required Processing Times and Capacity

Operations	*Time (hrs.) required per item of*			*Total available hours per month*
	X	*Y*	*Z*	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180

Profit (Rs.) per unit

Product:	X	Y	Z
Profit (Rs.)/unit:	10	15	8

Find out the product-mix to maximize profit. **[C.A. (Nov.) 89]**

Solution. Let x, y and z denote the number of units produced per month for the products X, Y and Z, respectively.

Minimum sales requirements give the constraints: $x \ge 10$, $y \ge 20$, $z \ge 30$, where $x, y, z \ge 0$.

Operations, processing times and capacity lead to the following constraints:

$$x + 2y + 2z \le 200 \ldots(i) \qquad 2x + y + z \le 220 \,..(ii) \qquad 3x + y + 2z \le 180 \,\ldots(iii)$$

The objective function is: Max. $\mathbf{P} = 10x + 15y + 8z$. Thus we have to solve the following problem:

Max. $\mathbf{P} = 10x + 15y + 8z$, *subject to* $x + 2y + 2z \le 200$, $2x + y + z \le 220$, $3x + y + 2z \le 180$, and $0 \le x \ge 10$, $0 \le y \ge 20$, $0 \le z \ge 30$.

Let us make the substitutions: $x = a + 10$, $y = b + 20$, $z = c + 30$, where $a, b, c \ge 0$.

Substituting these values in the objective function and constraints (i), (ii) and (iii), the problem becomes:

$$\textit{Max. } \mathbf{P} = 10a + 15b + 8c + 640, \ \textit{subject to,}$$
$$(a + 10) + 2\,(b + 20) + 2\,(c + 30) \le 200$$
$$2(a + 10) + (b + 20) + (c + 30) \le 220$$
$$3(a + 10) + (b + 20) + 2\,(c + 30) \le 180$$

where $a \ge 0, b \ge 0, c \ge 0$.

Solving this problem by simplex method we get the solution : $a = 10$, $b = 40$ and $c = 0$. Substituting these values, we find the original values :

$x = 10 + 10 = 20$, $y = 40 + 20 = 60$, $z = 0 + 30 = 30$, and the maximum value of objective function is given by $P =$ Rs. 1340.

The optimal product mix is to produce 20 units of X, 60 units of Y, and 30 units of Z to get a maximum profit of Rs. 1340.

Example 9. *Nooh's Boats makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table:*

Input	Row Boat	Canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1.260 hrs.
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg.)	2	4	3	396 Kg.
Selling price (in Rs.)	4,000	2,000	5,000	

(a) Formulate the above as a linear programming problem.

(b) Solve it by simplex method. From the optimal table of the solved linear programming problem, answer the following questions :

(c) How many boats of each type will be produced and what will be the resulting revenue ?

(d) Which, if any, of the resources are not fully utilized ? If so, how much of spare capacity is left ?

(e) How much wood will be used to make all of the boats given in the optimal solution ? **[C.A. (Nov.) 93]**

Solution. (a) Let x_1, x_2 and x_3 be the number of *Row Boats, Canoe* and *Keyak* made every month. The linear programming model can be formulated as follows :

Max. Revenue $z = 4{,}000x_1 + 2{,}000x_2 + 5{,}000x_3$, *subject to*

$12x_1 + 7x_2 + 9x_3 \le 1260,\ 22x_1 + 18x_2 + 16x_3 \le 19008,\ 2x_1 + 4x_2 + 3x_3 \le 396$ and $x_1, x_2, x_3, \ge 0$.

(b) Adding slack variables s_1, s_2, s_3, the above formulated problem becomes

Max. $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$, *subject to :*

$12x_1 + 7x_2 + 9x_3 + s_1 = 1260,\ 22x_1 + 18x_2 + 16x_3 + s_2 = 19008,\ 2x_1 + 4x_2 + 3x_3 + s_3 = 396$, and $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$.

The starting solution and subsequent simplex tables are given below :

		$c_j \rightarrow$	4000	2000	5000	0	0	0	
Basic Variables	Prog. C_B	Qty X_B	X_1	X_2	X_3	S_1	S_2	S_3	Replacement Ratio Min (X_B/X_K)
s_1	0	1,260	12	7	9	1	0	0	1260/9
s_2	0	19,008	22	18	16	0	1	0	19008/16
s_3	0	396	2	4	[3]	0	0	1	396/3 ←
	$z = 0$		−4000	−2000	−5000↑	0	0	0↓	← Δ_j (NER)
s_1	0	72	[6]	−5	0	1	0	−3	−12 ←
s_2	0	16,896	34/3	−10/.3	0	0	1	−16/3	1491
x_3	5000	132	2/3	4/3	1	0	0	1/3	198
	$z = 660000$		−2000/3↑	14,000/3	0	0↓	0	5000/3	← Δ_j
x_1	4000	12	1	−5/6	0	1/6	0	−1/2	
s_2	0	16,760	0	55/9	0	−17/9	1	1/3	
x_3	5000	124	0	17/9	1	−1/9	0	2/3	
	$z = 6{,}68{,}000$		0	37,000/9	0	1000/9	0	4000/3	← Δ_j

Since all $\Delta_j \ge 0$, the optimal solution is given by $x_1 = 12$, $x_2 = 0$ and $x_3 = 124$.

(c) The company should produce 12 *Row boats* and *124 Kayak boats* only. The maximum revenue will be Rs. 6,68,000.

(d) Wood is not fully utilized. Its share capacity is 16,760 board feet.

(e) The total wood used to make all of the boats given by the optimum solution is

$$= 22 \times 12 + 16 \times 124 = 2{,}248 \text{ board feet.}$$

EXAMINATION PROBLEMS

Solve the following problems by simplex method :

1. Max. $z = 5x_1 + 3x_2$, subject to
   $3x_1 + 5x_2 \le 15$
   $5x_1 + 2x_2 \le 10$
   $x_1, x_2 \ge 0$.
   [Ans. $x_1 = 20/19$, $x_2 = 45/19$, Max. $z = 235/19$**]**

2. Max. $z = 7x_1 + 5x_2$, subject to
   $-x_1 - 2x_2 \ge -6$
   $4x_1 + 3x_2 \le 12$,
   $x_1, x_2 \ge 0$ **[Meerut (IPM) 91]**
   [Ans. $x_1 = 3$, $x_2 = 0$, Max. $z = 21$**]**

3. Max. $z = 5x_1 + 7x_2$, subject to
$x_1 + x_2 \le 4$
$3x_1 - 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
and $x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 4$, *max.* $z = 28$]

4. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 40$
$x_1 + x_2 \le 24$
$2x_1 + 3x_2 \le 60$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 16, x_2 = 8, z^* = 64$]

5. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 5$
$x_1 + x_2 \le 3$
$x_1, x_2 \ge 0$
and $x_1, x_2 \ge 0$. **[I.A.S (Main) 91]**
[**Ans.** $x_1 = 6, x_2 = 12, z^* = 60$]

6. Max. $z = 2x_1 + 4x_2$, subject to
$2x_1 + 3x_2 \le 48$
$x_1 + 3x_2 \le 42$
$x_1 + x_2 \le 21$
and $x_1, x_2 \ge 0$
[**Ans.** Solution is unbounded]

7. Max. $z = 3x_1 + 4x_2$, subject to
$x_1 - x_2 \le 1$
$-x_1 + x_2 \le 2$
$x_1, x_2 \ge 0$.
[**Ans.** Sol. is unbounded]

8. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 10$
$x_1 + 3x_2 \le 6$
$x_1, x_2 > 0$
[**Ans.** $x_1 = 24/5, x_2 = 2/5, z^* = 76/5$]

9. Max. $z = 2x_1 + 5x_2$, subject to
$x_1 + 3x_2 \le 3$
$3x_1 + 2x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 0, z^* = 4$]

10. Max. $z = 3x_1 + 5x_2$, subject to
$3x_1 + 2x_2 \le 18$
$x_1 \le 4$
$x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 6, z^* = 36$]

11. Max. $z = 2x_1 + x_2$, subject to
$x_1 + 2x_2 \le 10$
$x_1 + x_2 \le 6$
$x_1 - x_2 \le 2$
$x_1 - 2x_2 \le 1$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 4, x_2 = 2, z^* = 10$]

12. Max. $z = 2x + 5y$, subject to
$x + y \le 600$
$0 \le x \le 400$
$0 \le y \le 300$
[**Ans.** Two iterations.
$x = 300, y = 300$, max $z = 2100$]

13. Max. $z = x_1 - x_2 + 3x_3$, subject to
$x_1 + x_2 + x_3 \le 10$
$2x_1 \quad - x_3 \le 2$
$2x_1 - 2x_2 + 3x_3 \le 0$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 6, x_3 = 4, z^* = 6$]

14. Max. $z = x_1 + x_2 + x_3$, subject to
$4x_1 + 5x_2 + 3x_3 \le 15$
$10x_1 + 7x_2 + x_3 \le 12$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 0, x_3 = 5, z^* = 5$]

15. Max. $z = 8x_1 + 19x_2 + 7x_3$,
subject to
$3x_1 + 4x_2 + x_3 \le 25$
$x_1 + 3x_2 + 3x_3 \le 50$
$x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 7/3, x_2 = 9, x_3 = 0$]

16. Max. $z = x_1 + x_2 + 3x_3$,
subject to
$3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0$.
[VTU (BE common) 2002]
[**Ans.** $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3$]

17. Max. $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$,
subject to
$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$
$2x_1 \quad + 2x_3 + x_4 \le 60$
$3x_1 + 3x_2 + x_3 + x_4 \le 80$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 280/13, x_2 = 0$,
$x_3 = 20/13, x_4 = 180/13$
$z^* = 2280/13$].

18. Max. $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$.
subject to
$x_1 + x_2 + x_3 + x_4 \le 15$
$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$
$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 50/7, x_2 = 0, x_3 = 55/7$,
$x_4 = 0, z^* = 695/7$]

19. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$,
subject to
$2x_1 + x_2 + 2x_3 + 3x_4 \le 12$,
$3x_1 + \quad 2x_3 + 2x_4 \le 20$,
$2x_1 + x_2 + 4x_3 \quad \le 16$,
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = x_3 = x_4 = 0, x_2 = 12$,
max. $z = 48$. On iteration only]

20. Max. $z = 5x_1 + 3x_2$,
subject to the constraints :
$x_1 + x_2 \le 2$
$5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 0$, max. $z = 10$,
one iteration only]

21. Max. $z = 8x_1 + 11x_2$
subject to the constraints :
$3x_1 + x_2 \le 7, x_1 + 3x_2 \le 8, x_1, x_2 \ge 0$.
[**Ans.** Two iterations.
$x_1 = 13/8, x_2 = 17/8$, max $z = 291/8$]

22. Max. $z = 10x_1 + x_2 + 2x_3$, subject to the constraints
$x_1 + x_2 - 3x_3 \le 10,\ 4x_1 + x_2 + x_3 \le 20,\ x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 5, x_2 = 0, x_3 = 0$, max. $z = 50$]

23. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints : $x_1 + 3x_2 + x_4 \le 4,\ 2x_1 + x_2 \le 3,\ x_2 + 4x_3 + x_4 \le 3;\ x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 1, x_2 = 1, x_3 = 1/2, x_4 = 0$, max. $z = 13/2$]

24. Max. $z = 10x_1 + 6x_2$, subject to the constraints :
$x_1 + x_2 \le 2,\ 2x_1 + x_2 \le 4,\ 3x_1 + 8x_2 \le 12$, and
$x_1, x_2 \ge 0$.
[**Ans.** One iteration only.
$x_1 = 2, x_2 = 0$, max. $z = 20$]

25. Max. $z = 107x_1 + x_2 + 2x_3$, subject to the constraints :
$14x_1 + x_2 - 6x_3 + 3x_4 = 7,\ 16x_1 + 1/2x_2 - 6x_3 \le 5,\ 3x_1 - x_2 - x_3 \le 0$.
$x_1, x_2, x_3, x_4 \ge 0$.
[**Hint.** Divide the first equation by 3 (coefficient of x_4) and then treat x_4 as the slack variable]. [**Ans.** Unbounded solution].

26. Explain the simplex method by carrying out one iteration in the following problem :
Max. $z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$, subject to the constraints :

$x_1 + 2x_2 + 2x_3 + x_4 = 8,\ 3x_1 + 4x_2 + x_3 + x_5 = 7$ and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$.

[**Ans.** One iteration only, $x_1 = x_2 = x_4 = 0,\ x_3 = 4,\ x_5 = 3$, max. $z = 15$]

27. Max. $z = 3x_1 + 2x_2 - 2x_3$
subject to the constraints:
$x_1 + 2x_2 + 2x_3 \leq 10$
$2x_1 + 4x_2 + 3x_3 \leq 15$
$x_1,\ x_2 < x_3 \geq 0$
[**Ans.** One iteration only.
$x_1 = 15/2,\ x_2 = x_3 = 0$, max. $z = 45/2$]

28. Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints:
$6x_1 + 5x_2 + 3x_3 \leq 52$
$6x_1 + 2x_2 + 5x_3 \leq 14$.
$x_1,\ x_2,\ x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = 0,\ x_2 = 7,\ x_3 = 0$, max. $z = 161$]

29. Max. $z = 7x_1 + x_2 + 2x_3$,
subject to the constraints:
$x_1 + x_2 - 2x_3 \leq 10$
$4x_1 + x_2 + x_3 \leq 20$
$x_1,\ x_2,\ x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = x_2 = 0,\ x_3 = 20$
max. $z = 40$]

30. Max. $R = 2x - 3y + z$
subject to the constraints:
$3x + 6y + z \leq 6$
$4x + 2y + z \leq 4$
$x - y + z \leq 3$
$x, y, z \geq 0$
[**Ans.** Two iterations. $x = 1/3$,
$y = 0,\ z = 8/3$, max. $R = 10/3$]

31. Max. $R = 2x + 4y + 3z$
subject to the constraints:
$3x + 4y + 2z \leq 60$
$2x + y + 2z \leq 40$
$x + 3y + 2z \leq 80$
$x, y, z \geq 0$.
[**Ans.** Two iterations. $x = 0,\ y = 20/3$,
$z = 50/7$, max. $R = 250/3$.]

32. Max. $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$
subject to the constraints:
$x_1 + x_4 + 6x_6 = 9$
$3x_1 + x_2 - 4x_3 + 2x_6 = 2$
$x_1 + 2x_3 + x_5 + 2x_6 = 6$
$x_i \geq 0,\ i = 1, 2, 3, 4, 5, 6$.
[**Ans.** One iteration. $x_1 = 2/3,\ x_2 = x_3 = 0$,
$x_4 = 25/3,\ x_5 = 16/3,\ x_6 = 0$, max $z = 43/3$]

33. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yield a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how masny acres should be allocated to each crop to maximize profit? **[Jammu Univ. (MBA) Feb. 96]**

[**Hint.** Formulation of the problem is:
Max. $z = 30x_1 + 40x_2 + 20x_3$, s.t.
$10x_1 + 12x_2 + 7x_3 \geq 10{,}000;\ 7x_1 + 10x_2 + 8x_3 \leq 8{,}000$
$x_1 + x_2 + x_3 \leq 1{,}000;\ x_1, x_2, x_3 \geq 0$.]

[**Ans.** Acreage for cam, wheat and soyabeans are 250, 625 and respectively with max. profit of Rs. 32,500]

5·13 ARTIFICIAL VARIABLE TECHNIQUE

5·13-1 Two Phase Method **[Garhwal 97; Kanpur (B.Sc.) 90; Rohil. 90]**

Linear programming problems, in which constraints may also have '≥' and '=' signs after ensuring that all b_i are ≥ 0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the ***artificial variable.*** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in ***Phase I*** of the solution, and ***Phase II*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called ***'Two Phase Simplex Method'*** due to **Dantzig, Orden and Wolfe.**

Remarks:

1. **The objective of Phase I is to search for a B.F.S. to the given problem It ends up either giving a B F.S. or indicating that the given L.P.P. has no feasible solution at all.**
2. **The B.F.S. obtained at the end of Phase 1 provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.**
3. **In Phase II, care must be taken to ensure that an artificial variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artifical variable happens to leave the basis, its column must be deleted from the simplex table altogether.**

Q. 1. Explain the term 'Artificial variable' and its use in linear programming.
2. What do you mean by two phase-method in linear programming problems, why it is used ?

This technique is well explained by the following example.

Example 10. *Solve the problem : Minimize* $z = x_1 + x_2$, *subject to* $2x_1 + x_2 \geq 4$, $x_1 + 7x_2 \geq 7$, *and* $x_1, x_2 \geq 0$. **[Delhi B.Sc. (Math.) 91, 88; Bharthidasan B.Sc. (Math.) 90; VTU (BE common) Aug. 2002]**

Solution. First convert the problem of minimization to maximization by writing the objective function as :

$$\text{Max } (-z) = -x_1 - x_2 \quad \text{or} \quad \text{Max. } z' = -x_1 - x_2, \text{ where } z' = -z.$$

Since all b_i's (4 and 7) are positive, the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become :

$$2x_1 + x_2 - x_3 = 4$$
$$x_1 + 7x_2 - x_4 = 7$$

But the basis matrix **B** would not be an identity matrix due to negative coefficients of x_3 and x_4. Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as

$$2x_1 + x_2 - x_3 + a_1 = 4$$
$$x_1 + 7x_2 - x_4 + a_2 = 7.$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (*Table 5·14*) where A_1 and A_2 denote the artificial column-vectors corresponding to a_1 and a_2, respectively.

Table 5·14

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	4	2	1	−1	0	1	0
a_2	7	1	[7]	0	−1	0	1
			↑	×	×	×	↓

Now remove each artificial column vector A_1 and A_2 from the basis matrix. To remove vector A_2 first, select the entering vector either X_1 or X_2, being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector X_2 to enter the basis matrix. It can be easily verified that if the vector A_2 is entered in place of X_1, the resulting solution will not be feasible. Thus transformed table (Table 5·15) is obtained.

Table 5·15

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	3	13/7	0	−1	[1/7]	1	−1/7
x_2	1	1/7	1	0	−1/7	0	1/7
					↑	↓	↓

(Delete column A_2 for ever at this stage)

This table gives the solution : $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $a_1 = 3$, $a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector A_2 again for re-entry into the basis matrix.

Similarly, remove A_1 from the basis matrix by introducing it in place of X_4 by the same method. Thus Table 5·16 is obtained.

Table 5·16

BASIC VARAIBLES	X_B	X_1	X_2	X_3	X_4	A_1
x_4	21	13	0	−7	1	7
x_2	4	2	1	−1	0	↓

(Delete column A_1 for ever at this stage)

This table gives the solution : $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21, a_1 = 0$. Since the artificial variable a_1 becomes zero (non-basic), so drop the corresponding column $\mathbf{A}_1$ from this table. Thus, the solution $(x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21)$ is the basic feasible solution and now usual simplex routine can be started to obtain the required optimal solution.

Phase II. Now in order to test the starting above solution for optimality, construct the starting simplex Table 5.17

Table 5·17

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio (X_B/X_1)
		$c_j \rightarrow$	-1	-1	0	0	
$\leftarrow x_4$	0	21	$\leftarrow$ [13]	0	7	1	21/13 $\leftarrow$
x_2	-1	4	2	1	-1	0	4/2
	$z' = C_B X_B = -4$		-1 $\uparrow$	0	1	0 $\downarrow$	$\leftarrow \Delta_j$

Compute $\Delta_1 = -1, \Delta_3 = 1$.

Key element 13 indicates that $\mathbf{X}_4$ should be removed from the basis matrix. Thus, by usual transformation method Table 5·18 is formed.

Table 5·18

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. RATIO COLUMN
		$c_j \rightarrow$	-1	-1	0	0	
$\rightarrow x_1$	-1	21/13	1	0	$-7/13$	1/13	
x_2	-1	10/13	0	1	1/13	$-2/13$	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Also, verify that

$$\Delta_3 = C_B X_3 - c_3 = (-1, -1)(-7/13, 1/13) = 6/13$$

$$\Delta_4 = C_B X_4 - c_4 = (-1, -1)(1/13, -2/13) = 1/13.$$

Since all $\Delta_j \geq 0$, the required optimal solution is :

$$x_1 = 21/13, x_2 = 10/13 \text{ and min. } z = 31/13 \text{ (because } z = -z').$$

5·13-2 Simple Way for Two-Phase Simplex Method

Phase I : Table 5·19

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	4	2	1	-1	0	1	0
$\leftarrow a_2$	7	1	[7] $\uparrow$	0	-1	0	1 $\downarrow$
$\leftarrow a_1$	3	13/7	0	-1	[1/7]	1	$-1/7$
$\rightarrow x_2$	1	1/7	1	0	$-1/7$ $\uparrow$	0 $\downarrow$	1/7
$\rightarrow x_4$	21	13	0	-7	1	7	×
x_2	4	2	1	-1	0	1	

Thus, initial basic feasible solution is : $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21$. Now start to improve this solution in Phase II by usual simplex method.

Note :

1. Remove the artificial vector $\mathbf{A}_2$ and insert it anywhere such that $\mathbf{X}_B$ remains feasible (≥ 0).
2. As soon as $\mathbf{A}_2$ is removed from the basis by matrix transformation or otherwise, delete $\mathbf{A}_2$ for ever.
3. Similar process is adopted to remove other artificial vectors one by one from the basis.
4. Purpose of introducing artificial vectors is only to provide an initial basic feasible solution to start with simplex method in Phase II. So, as soon as the artificial variables become non-basic (*i.e.* zero), delete artificial vectors to enter Phase II.
5. Then, start Phase II, which is exactly the same as original simplex method.

Phase II. Table 5·20

		$c_j \rightarrow$	-1	-1	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. RATIO (X_B/X_k)
$\leftarrow x_4$	0	21	$\boxed{13}$ ↑	0	7	1	$21/13 \leftarrow$
x_2	-1	4	2	1	-1	0	$4/2$
	$z' = -4$		-1^* $\uparrow$	0	1	0 $\downarrow$	$\leftarrow \Delta_j$
$\rightarrow x_1$	-1	$21/13$	1	0	$-7/13$	$1/13$	
x_2	-1	$10/13$	0	1	$1/10$	$2/13$	
	$z' = -31/13$		0	0	$6/13$	$1/13$	$\leftarrow \Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = 21/13$, $x_2 = 10/13$, max. $z = 31/13$.

5.13-3 *Alternative Approach* of Two-phase Simplex Method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise :

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to *Phase–II.*

(iii) Max $z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to *Phase–II.*

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of *Phase–I*, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of *Phase–I* are removed.

Q. 1. What are artificial variables ? Why do we need them ? Describe briefly the two-phase method of solving a L.P. problem with artificial variables. **[Meerut M.Sc. (Math.) 93]**

2. What do you mean by two phase method for solving a given L.P.P. ? Why is it used ?

3. Explain steps in solving a linear programming problem by two-phase method.

The following examples will make the *alternative* two-phase method clear.

Example 11. *Use two-phase simplex method to solve the problem : Minimize* $z = x_1 - 2x_2 - 3x_3$, *subject to the constraints* :$-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, *and* $x_1, x_2, x_3 \geq 0$, **[Meerut (Maths.) 91]**

Solution. First convert the objective function into maximization form :

$$\text{Max } z' = -x_1 + 2x_2 + 3x_3, \text{ where } z' = -z.$$

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 \qquad = 2$$
$$2x_1 + 3x_2 + 4x_3 \qquad + a_2 = 1$$
$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I. Auxiliary L.P. problem is : *Max.* $z'^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$ *subject to above given constraints.*

The following solution table is obtained for auxiliary problem.

Table 5·21

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	A_1	A_2	MIN. RATIO (X_B/X_k)
		$c_j \rightarrow$	0	0	0	−1	−1	
a_1	−1	2	−2	1	3	1	0	2/3
$\leftarrow a_2$	−1	1	2	3	[4] ↑	0	1	1/4 ←
	$z'^* = -3$		0	−4	-7^* ↑	0	0 ↓	$\leftarrow \Delta_j$
a_1	−1	5/4	−7/2	−5/4	0	1	−3/4	
$\rightarrow x_3$	0	1/4	1/2	3/4	1	0	1/4	
	$z'^* = -5/4$		7/4	5/4	0	0	3/4	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time max. z'^* is negative and the artificial variable a_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible solution. Here there is no need to enter *Phase II.*

Example 12. *Use two-phase simplex method to solve the problem :*

Minimize $z = 15/2\ x_1 - 3x_2$, *subject to the constraints* :

$$3x_1 - x_2 - x_3 \geq 3, \quad x_1 - x_2 + x_3 \geq 2, \ \textit{and}\ x_1, x_2, x_3 \geq 0.$$

Solution. Convert the objective function into the maximization form : Maximize. $z' = -15/2\ x_1 + 3x_2$.

Introducing the surplus variables $x_4 \geq 0$ and $x_5 \geq 0$, and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints of the given problem become

$$3x_1 - x_2 - x_3 - x_4 \qquad + a_1 \qquad = 3$$
$$x_1 - x_2 + x_3 \qquad - x_5 \qquad + a_2 = 2$$
$$x_1,\ x_2,\ x_3,\ x_4, a_1, a_2 \geq 0.$$

Phase I. Assigning a cost − 1 to artificial variables a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes : Max. $z'^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 - 1a_1 - 1a_2$, subject to the above given constraints.

Now apply simplex method in usual manner, (see *Table 5·22*).

Phase I : Table 5·22

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	MIN RATIO (X_B/X_k)
		$c_j \rightarrow$	0	0	0	0	0	−1	−1	
$\leftarrow a_1$	−1	3	[3]	−1	−1	−1	0	1	0	3/3 ←
a_2	−1	2	1	−1	1	0	−1	0	1	2/1
	$z'^* = -5$		-4^* ↑	2	0	1	1	0 ↓	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	1	1	−1/3	−1/3	−1/3	0	1/3	0	—
$\leftarrow a_2$	−1	1	0	−2/3	[4/3]	1/3	−1	1/3	1	3/4 ←
	$z'^* = -1$		0	2/3	$-4/3^*$ ↑	−1/3	1	2/3	0 ↓	$\leftarrow \Delta_j$
x_1	0	5/4	1	−1/2	0	−1/4	−1/4	1/4	1/4	
x_3	0	3/4	0	−1/2	1	1/4	−3/4	1/4	3/4	
	$z'^* = 0$		0	0	0	0	0	1	1	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to problem has been attained.

Phase 2. In this phase, now consider the actual costs associated with the original variables, function thus becomes : Max. $z' = -15/2\,x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

Now apply simplex method in the usual manner.

Phase 2 : Table 5·23

		$c_j \rightarrow$	$-15/2$	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_k)
x_1	$-15/2$	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	
x_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	
		$z' = -75/8$	0	$3/4$	0	$15/8$	$15/8$	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been attained.

Hence optimum solution is : $x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$, min $z = 75/8$.

EXAMINATION PROBLEMS

Solve the following LP problems by two-phase method:

1. Max. $z = 3x_1 - x_2$
subject to the constraints :
$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 2$
$x_2 \leq 4$
and $x_1, x_2 \geq 0$.
[**Ans.** $x_1 = 2$, $x_2 = 0$ Max $z = 6$]

2. Max. $z = 5x_1 + 8x_2$
subject to the constraints :
$3x_1 + 2x_2 \geq 3$
$x_1 + 4x_2 \geq 4$
$x_1 + x_2 \leq 5$
and $x_1, x_2 \geq 0$.
[**Ans.** $x_1 = 0$, $x_2 = 5$, max. $z = 40$]

3. Max $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$
with the conditions :
$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$
$2x_1 + x_2 + x_3 + 5x_4 \leq 4$
$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$
$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$
$x_i\,(i = 1, 2, 3, 4) \geq 0$
[**Ans.** $x_1 = 1.2$, $x_2 = 0$, $x_3 = 0.9$
$x_4 = 0$, max. $z = 19.8$]

4. Minimize $z = x_1 - 2x_2 - 3x_3$, subject to
$-2x_1 + x_2 + 3x_3 = 2$
$2x_1 + 3x_2 + 4x_3 = 1$,
$x_j \geq 0$, $j = 1, 2, 3$.
[**Ans.** Here all $\Delta_j \geq 0$, but at the same time artificial variable a_1 appears in the basis. Hence the given LPP has no feasible solution]

5. Max. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
subject to
$4x_1 + 5x_2 + x_3 - 3x_4 = 5$
$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$
$x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6$
$x_1, x_2, x_3 \geq 0$
[**Ans.** No solution]

6. Max $z = 5x_1 - 2x_2 + 3x_3$
subject to
$2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1, x_2, x_3, x_4 \geq 0$.
[AIMS (BE Ind.) Bang. 2002]
[**Ans.** $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$,
max. $z = 85/3$]

7. Max. $z = 2x_1 + 3x_2 + 5x_3$,
subject to the constraints :
$3x_1 + 10x_2 + 5x_3 \leq 15$,
$x_1 + 2x_2 + x_3 \geq 4$,
$33x_1 - 10x_2 + 9x_3 \leq 33$,
$x_1, x_2, x_3 \geq 0$.
[**Ans.** There does not exist any feasible solution, because artificial variable is not removed in the problem]

8. Max. $500\,x_1 + 1400\,x_2 + 900x_3$,
subject to,
$x_1 + x_2 + x_3 = 100$
$12x_1 + 35x_2 + 15x_3 \geq 25$
$8x_1 + 3x_2 + 4x_3 \geq 6$;
$x_1, x_2, x_3 \geq 0$. **[Meerut (MA) 93]**

9. A firm has an advertising budget of Rs. 7,20,000. It wishes to allocate this budget to two media : magazines and televisions, so that total exposure is maximized. Each page of magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of magazine advertising costs Rs. 9,000 and each spot on television costs Rs. 12,000. An additional condition that the firm has specified is that at least two pages of magazine advertising be used and at least 3 spots on television. Determine the optimum media-mix for this firm.
[Delhi (MBA) 97]
[**Hint.** The problem is :
Max. $z = 60{,}000x_1 + 12{,}000x_2$ s.t.
$9{,}000x_1 + 12{,}000x_2 \leq 7{,}20{,}000$, $x_1 \geq 2$, $x_2 \geq 3$, $x_1, x_2 \geq 0$,
where x_1 = no. of pages of magazine
x_2 = no. of spots on television]
[**Ans.** $x_1 = 2$, $x_2 = 58{\cdot}5$ and max. z = 7,14,000]

5.13-4 Big-M-Method (Charne's Penalty Method) [Kanpur (B.Sc.) 92, 91]

Computational steps of Big-M-method are as stated below :

Step 1. Express the problem in the standard form.

Step 2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type (≥) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M>0$.

Step 3. In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Q. 1. Explain the use of Big-M-method in solving L.P.P. What are its characteristics ?

Example 13. *Solve by using big-M method the following linear programming problem* :

$$\text{Max. } z=-2x_1-x_2,\ \textit{subject to } 3x_1+x_2=3,\ 4x_1+3x_2\geq 6,\ x_1+2x_2\leq 4,\ \textit{and } x_1, x_2\geq 0.$$

Solution.

Step 1. Introducing slack, surplus and artificial variables, the system of constraint equations become :

$$\begin{aligned} 3x_1+x_2 \quad\quad + a_1 \quad &= 3\\ 4x_1+3x_2-x_3 \quad\quad + a_2 &= 6\\ x_1+2x_2+x_4 \quad\quad &= 4 \end{aligned}$$

which can be written in the matrix form as :

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4 & \mathbf{A}_1 & \mathbf{A}_2\\ 3 & 1 & 0 & 0 & 1 & 0\\ 4 & 3 & -1 & 0 & 0 & 1\\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ a_1\\ a_2\end{bmatrix}=\begin{bmatrix}3\\6\\4\end{bmatrix}$$

Step 2. Assigning the large negative price $-M$ to the artificial variables a_1 and a_2, the objective function becomes : Max. $z=-2x_1-x_2+0x_3+0x_4-Ma_1-Ma_2$.

Step 3. Construct starting simplex table (Table 5·24)

Starting Simplex Table 5·24

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X_1})$
$\leftarrow a_1$	$-M$	3	[3]	1	0	0	1	0	$3/3 \leftarrow$
a_2	$-M$	6	4	3	-1	0	0	1	6/4
x_4	0	4	1	2	0	1	0	0	4/1
		$z=-9M$	$(2-7M)$ ↑	$(1-4M)$	M	0	0 ↓	0	$\leftarrow \Delta_j$

To apply optimality test, compute

$\Delta_1=\mathbf{C_B X_1}-c_1=(-M,-M,0)(3,4,1)-(-2)=2+(-3M-4M+0)=2-7M$

$\Delta_2=\mathbf{C_B X_2}-c_2=(-M,-M,0)(1,3,2)-(-1)=1+(-M-3M+0)=1-4M$

$\Delta_3=\mathbf{C_B X_3}-c_3=(-M,-M,0)(0,-1,0)+0=M$

$\therefore\ \Delta_k=\min[\Delta_1,\Delta_2,\Delta_3]=\min[2-7M,1-4M,M]=\Delta_1$. Therefore, $\mathbf{X_1}$ will be entered.

Using minimum ratio rule, find the key element 3 which indicates that $\mathbf{A_1}$ should be removed. Now the transformed table (*Table 5·25*) is obtained in usual manner.

First Improved Table 5.25

BASIC VARIABLES	C_B	X_B	X_1 (−2)	X_2 (−1)	X_3 (0)	X_4 (0)	A_1 (−M)	A_2 (−M)	MIN RATIO (X_B/X_k)
$\to x_1$	−2	1	1	1/3	0	0	1/3	0	$1/\frac{1}{3}$
$\leftarrow a_2$	−M	2	0	[5/3] ↑	−1	0	−4/3	1	$2/\frac{5}{3} \leftarrow$
x_4	0	3	0	5/3	0	1	−1/3	0	$3/\frac{5}{3}$
	$z = -2 - 2M$		0	$(1-5M)/3$ ↑	M	0	$(-2+7M)/3$	0 ↓	$\leftarrow \Delta_j$

Again compute, $\Delta_2 = C_B X_2 - c_2 = (-2, -M, 0)(1/3, 5/3, 5/3) + 1 = (1-5M)/3$, and similarly, $\Delta_3 = M$, $\Delta_5 = (-2 + 7M)/3$.

Since *minimum Δ_j rule and minimum ratio rule* decide the key element 5/3, so enter X_2 and remove A_2. Therefore, the second improved table (Table 5.26) is formed.

Table 5.26

BASIC VARIABLES	C_B	X_B	X_1 (−2)	X_2 (−1)	X_3 (0)	X_4 (0)	A_1 (−M)	A_2 (−M)	MIN. RATIO
x_1	−2	3/5	1	0	1/5	0	3/5	−1/5	
x_2	−1	6/5	0	1	−3/5	0	−4/5	3/5	
x_4	0	1	0	0	1	1	1	−1	
	$z = C_B X_B = -12/5$		0	0	1/5	0	$M - 2/5$	$M - 1/5$	$\leftarrow \Delta_j \geq 0$

To test the solution for optimality, compute

$$\Delta_3 = C_B X_3 - c_3 = (-2, -1, 0)(1/5, -3/5, 1) - 0 = 1/5$$
$$\Delta_5 = C_B A_2 - c_5 = (-2, -1, 0)(3/5, -4/5, 1) + M = M - 2/5$$
$$\Delta_6 = C_B A_2 - c_6 = (-2, -1, 0)(-1/5, -3/5, -1) + M = M - 1/5.$$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive. Consequently, the optimal solution is : $x_1 = 3/5$, $x_2 = 6/5$, max $z = -12/5$.

Example 14. *Solve the following problem by Big-M-method : Max.* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to :* $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

[IAS (Maths.) 95; Kanpur (B.Sc.) 92; Karala (B.Sc.) 91; Meerut (B.Sc.) 90]

Solution. Since the constraints of the given problem are equations, introduce the artificial variables $a_1 \geq 0$, $a_2 \geq 0$. The problem thus becomes :

Max. $z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$, subject to the constraints :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad\quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad\quad &= 10 \\ \text{and } x_1, x_2, x_3, x_4, a_1, a_2 \quad &\geq 0. \end{aligned}$$

Now applying the usual simplex method, the solution is obtained as given in the *Table 5·27.*

Table 5·27 (Example 14)

		$c_j \rightarrow$	1	2	3	-1	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_3)
a_1	$-M$	15	1	2	3	0	1	0	15/3
$\leftarrow a_2$	$-M$	20	2	1	[5]	0	0	1	20/5 ←
x_4	-1	10	1	2	1	1	0	0	10/1
	$z = (-35M - 10)$		$(-3M-2)$	$(-3M-2)$	$(-8M-4)$ ↑	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	3	$-1/5$	[7/5]	0	0	1	×	$\frac{3}{7/5}$ ←
$\rightarrow x_3$	3	4	2/5	1/5	1	0	0	×	$\frac{4}{1/5}$
x_4	-1	6	3/5	9/5	0	1	0	×	$\frac{6}{9/5}$
	$z = (-3M + 6)$		$(M-2)/5$	$-(7M-16)/5$ ↑	0	0	0 ↓	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	15/7	$-1/7$	1	0	0	×	×	—
x_3	3	25/7	3/7	0	1	0	×	×	25/3
$\leftarrow x_4$	-1	15/7	[6/7]	0	0	1	×	×	15/6 ←
	$z = 90/7$		$-6/7$* ↑	0	0	0 ↓	×	×	$\leftarrow \Delta_j$
x_2	2	15/6	0	1	0	1/6	×	×	
x_3	3	15/6	0	0	1	3/6	×	×	
$\rightarrow x_1$	1	15/6	1	0	0	7/6	×	×	
	$z = 15$		0	0	0	75/36	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained as :

$$x_1 = x_2 = x_3 = \frac{15}{6} = \frac{5}{2}, \text{ max } z = 15 .$$

Example 15. *Use penalty (Big-M) method to maximize* : $z = 3x_1 - x_2$ *subject to the constraints :*

$$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 3, x_2 \leq 4, \text{and } x_1, x_2 \geq 0 .$$

Solution. By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack variables $x_4 \geq 0$, $x_5 \geq 0$, the problem becomes : Max. $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad + a_1 &= 2 \\ x_1 + 3x_2 \quad + x_4 \quad &= 3 \\ x_2 \quad + x_5 &= 4 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0 . \end{aligned}$$

In matrix form,
$$\begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Now the solution is obtained as given in Table 5.28

Table 5.28 [Example 15]

BASIC VARIABLES	C_B	X_B	X_1 (3)	X_2 (−1)	X_3 (0)	X_4 (0)	X_5 (0)	A_1 (−M)	MIN. RATIO (X_B/X_k)
$\leftarrow a_1$	$-M$	2	[2]	1	1	0	0	1	2/2 ←
x_4	0	3	1	3	0	1	0	0	3/1
x_5	0	4	0	1	0	0	1	0	—
	$z=-2M$		$(-2M-3)$ ↑	$-M+1$	M	0	0	0 ↓	
$\rightarrow x_1$	3	1	1	1/2	−1/2	0	0	×	—
$\leftarrow x_4$	0	2	0	5/2	[1/2]	1	0	×	$2/\frac{1}{2}$ ←
x_5	0	4	0	1	0	0	1	×	—
	$z=3$		0	5/2	$(-3/2)$ ↑	0 ↓	0	×	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1	0	×	
$\rightarrow x_3$	0	4	0	5	1	2	0	×	
x_5	0	4	0	1	0	0	1	×	
	$z=9$		0	10	0	3	0	×	$\leftarrow \Delta_j \geq 0$

Thus the optimum solution is obtained as : $x_1 = 3$, $x_2 = 0$, max. $z = 9$.

Example 16. ***(Unrestricted Variables)***

(a) Maximize $z = 8x_2$*, subject to the constraints :* $x_1 - x_2 \geq 0, 2x_1 + 3x_2 \leq -6$ *and* x_1 , x_2 *are unrestricted.* **[Meerut (Maths) 93]**

(b) Solve the LPP : Max $z = 4x_1 + 6x_2$*, subject to :* $x_1 - 2x_2 \geq -4$, $2x_1 + 44x_2 \geq 12$, $x_1 + 3x_2 \geq 9$ *and* x_1, x_2 *are unrestricted.* **[Meerut (Maths.) 96]**

Solution. (a) In this problem, the variables x_1 and x_2 are unrestricted in sign, *i.e.* x_1 and x_2 may be + ive, –ive or zero. But, the simplex method can be used only when the variables are non-negative (≥ 0) . This difficulty can be immediately removed by using the transformation :

$$x_1 = x_1' - x_1'' \text{ and } x_2 = x_2' - x_2'' \text{ such that } x_1' \geq 0,\ x_1'' \geq 0,\ x_2' \geq 0,\ x_2'' \geq 0 .$$

Therefore, the given problem becomes : *maximize* $z = 8x_2' - 8x_2''$*, subject to the constraints :*

$$(x_1' - x_1'') - (x_2' - x_2'') \geq 0$$
$$-2(x_1' - x_1'') - 3(x_2' - x_2'') \geq 6$$
$$x_1', x_1'', x_2', x_2'' \geq 0 .$$

Now introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the given problem becomes : Max. $z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0x_3 + 0x_4 - Ma_1 - Ma_2$, subject to :

$$x_1' - x_1'' - x_2' + x_2'' - x_3 + a_1 = 0$$
$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - x_4 + a_2 = 6$$
$$x_1', x_1'', x_2', x_2'', x_3, x_4, a_1, a_2 \geq 0 .$$

Table 5.29

BASIC VAR.	C_B	X_B	X_1' (0)	X_1'' (0)	X_2' (8)	X_2'' (−8)	X_3 (0)	X_4 (0)	A_1 (−M)	A_2 (−M)	MIN RATIO (X_B/X_k)
$\leftarrow a_1$	$-M$	0	1	−1	−1	[1]	−1	0	1	0	0 ←
a_2	$-M$	6	−2	2	−3	3	0	−1	0	1	6/3
	$z=-6M$		M	$-M$	$(4M-8)$	$(-4M+8)$ ↑	M	M	0 ↓	1	$\leftarrow \Delta_j$
$\rightarrow x_2'$	−8	0	1	−1	−1	1	−1	0	×	0	—
$\leftarrow a_2$	$-M$	6	−5	[5]	0	0	3	−1	×	1	6/5 ←
	$z=-6M$		$(5M-8)$	$(-5M+8)$ ↑	0	0	$(-3M+8)$	M	×	0 ↓	$\leftarrow \Delta_j$
x_2''	−8	6/5	0	0	−1	1	−2/5	1/5	×	×	
$\rightarrow x_1''$	0	6/5	−1	1	0	0	3/5	−1/5	×	×	
	$z=-48/5$		0	0	0	0	16/5	8/5	×	×	$\leftarrow \Delta_j \geq 0$

Remember that the coefficients of slack or surplus variables in the objective function are always zero and the coefficient of artificial variables is taken a *largest negative quantity* $-M$ *where* $M > 0$.

Applying the simplex method in the usual manner, the solution is obtained as given in *Table 5·29*.

Since all $\Delta_j \geq 0$, an optimum solution is obtained as : $x_1' = 0$, $x_1'' = 6/5$, $x_2' = 0$, $x_2'' = 6/5$.

Since $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$, transforming the solution to original variables, we get

$$x_1 = 0 - 6/5 = -6/5,\ x_2 = 0 - 6/5 = -6/5,\ \text{max. } z = -48/5.$$

(b) Solve as (a).

Note. Whenever the rarge of a variable is not given in the problem, it should be understood that such variable is unrestricted in sign.

Example 17. (Imp.) *Maximize* $z = 4x_1 + 5x_2 - 3x_3 + 50$, *subject to the constraints :*

$$x_1 + x_2 + x_3 = 10 \quad \text{...(i)}$$
$$x_1 - x_2 \geq 1 \quad \text{....(ii)}$$
$$2x_1 + 3x_2 + x_3 \leq 40 \quad \text{...(iii)}$$
$$x_1, x_2, x_3 \geq 0.$$

[Meerut (Maths.) 97 P]

Solution. *If any constant is included in the objective function (like 50 here) it should be deleted in the beginning and finally adjusted in optimum value of z and, if there is an equality in the constraints, then one variable can be eliminated from the inequalities with* ≤ or ≥ *sign. (Note)*

Subtracting *(i)* from *(iii)* with a view to eliminate x_3 from (iii) and retaining x_3 in (i) to work as a slack variable, the restrictions are modified as follows :

$$x_1 + x_2 + x_3 = 10, x_1 - x_2 \geq 1, x_1 + 2x_2 \leq 30, \text{and } x_1, x_2, x_3 \geq 0.$$

Now introducing the slack, surplus and artificial variables, the problem becomes :

Max. $z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$, subject to the constraints :

$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 - x_4 + a_1 = 1$$
$$x_1 + 2x_2 + x_5 = 30$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0.$$

Applying the usual simplex method, the solution is obtained as given in Table 5·30.

Table 5·30 [Example 17]

		$c_j \rightarrow$	4	5	−3	0	−M	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	X_5	MIN. RATIO (X_B/X_k)
x_3	−3	10	1	1	1	0	0	0	10/1
$\leftarrow a_1$	−M	1	[1]	−1	0	−1	1	0	1/1 ←
x_5	0	30	1	2	0	0	0	1	30/1
	$z = -30 - M$		−7−M ↑	−8+M	0	M	0 ↓	0	$\leftarrow \Delta_j$
$\leftarrow x_3$	−3	9	0	[2]	1	1	×	0	9/2 ←
$\rightarrow x_1$	4	1	1	−1	0	−1	×	0	—
x_5	0	29	0	3	0	1	×	1	29/3
	$z = 9/2$		0	−15* ↑	0 ↓	−7	×	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	5	9/2	0	1	1/2	1/2	×	0	
x_1	4	11/2	1	0	1/2	−1/2	×	0	
x_5	0	31/2	0	0	−3/2	−1/2	×	1	
	$z = 89/2$		0	0	15/2	1/2	×	0	$\leftarrow \Delta_j \geq 0$

Hence the solution is : $x_1 = 11/2, x_2 = 9/12$, max. $z = 89/2 + 50 = 189/2$.

Example 18. *Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.* **[Bharthidasan B.Sc. (Math.) 90]**

Solution. Let x_1 grams of food X and x_2 grams of food Y be purchased. Then the problem can be formulated as : Minimize $z = 12x_1 + 20x_2$, subject to the constraints : $6x_1 + 8x_2 \geq 100$, $7x_1 + 12x_2 \geq 120$, and $x_1, x_2 \geq 0$.

Introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints become :

$$6x_1 + 8x_2 - x_3 \quad + a_1 \quad = 100$$
$$7x_1 + 12x_2 \quad - x_4 \quad + a_2 = 120.$$

Objective function becomes :

$$\text{Max. } z' = -12x_1 - 20x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2, \text{ where } z' = -z.$$

Now proceeding by usual simplex method, the solution is obtained as given in Table 5.31.

Table 5.31 [Example 18]

	$c_j \rightarrow$		-12	-20	0	0	$-M$	$-M$	
BASIC VAR.	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	MIN RATIO (X_B/X_k)
a_1	$-M$	100	6	8 ↑	-1	0	1	0	100/8
$\leftarrow a_2$	$-M$	120	7	[12]	0	-1	0	1	120/12 ←
	$z' = -220M$		$(-13M + 12)$	$(-20M + 20)$ ↑	M	M	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	20	[4/3]	0	-1	2/3	1	×	60/4 ←
$\rightarrow x_2$	-20	10	7/12	1	0	$-11/2$	0	×	120/7
	$z' = -20M - 200$		$-(4M - 1)/3$ ↑	0	M	$(-2M + 5)/3$	0 ↓	×	$\leftarrow \Delta_j$
$\rightarrow x_1$	-12	15	1	0	$-3/4$	1/2	×	×	
x_2	-20	5/4	0	1	7/16	$-3/4$	×	×	
	$z' = -205$		0	0	1/4	9	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal solution is attained. Hence the optimal solution is :

$$x_1 = 15, x_2 = 5/4, \max z = -(-205) = 205.$$

Hence 15 grams of food X and 5/4 grams of food Y should be the required product-mix with minimum cost of Rs. 205.

5·14 DISADVANTAGES OF BIG-M-METHOD OVER TWO-PHASE METHOD

1. Although big-M method can always be used to check the existence of a feasbile solution, it may be computationally inconvenient because of the manipulation of the constant M. On the other hand, two-phase method eliminates the constant M from calculations.
2. Another difficulty arises specially when the problem is to be solved on a digital computer. To use a digital computer, M must be assigned some numerical value which is much larger than the values $c_1, c_2, \ldots,$ in the objective function. But, a computer has only a fixed number of digits.

Q. 1. What is an artificial variable and why it is necessary to introduce it ? Describe the two phase process of solving an L.P.P. by simplex method. **[Delhi B.Sc. (Maths.) 90]**

2. Why is an artificial vector which leaves the basis once never considered again for re-entry into the basis ? **[Delhi B.Sc. (Math.) 91]**

3. In the two-phase method explain when phase 1 terminates.

4. Optimality criteria being satisfied; state what is indicated by each of the following :
(i) One or more artificial vectors are in the basis at zero level.
(ii) One or more artificial vectors are in the basis at positive level. **[Delhi B.Sc. (Math.) 91]**

EXAMINATION PROBLEMS

Solve the following LP problems using Charne's Big-M (Penalty) method:

1. Min. $z = 2x_1 + 9x_2 + x_3$, subject to the constraints : $x_1 + 4x_2 + 2x_3 \geq 5$, $3x_1 + x_2 + 2x_3 \geq 4$, $x_1, x_2, x_3 \geq 0$.
[**Ans.** $x_1 = 0$, $x_2 = 0$, $x_3 = 5/2$, min. $z = 5/2$].

2. Max. $R = 5x - 2y - z$, subject to the constraints : $2x + 2y - z \geq 2$, $3x - 4y \geq 3$, $y + 3z \geq 5$, $x, y, z \geq 0$.
[**Ans.** $x = 13/9$, $y = 1/3$, $z = 14/9$, max. $R = 5$]

3. Min. $z = x_1 + x_2 + 3x_3$, subject to $3x_1 + 2x_2 + x_3 \leq 3$, $2x_1 + x_2 + 2x_3 \geq 3$ $x_1, x_2, x_3 \geq 0$.
[**Ans.** $x_1 = 3/4$, $x_2 = 0$, $x_3 = 3/4$, min. $z = 3$]

4. Min. $z = 4x_1 + 8x_2 + 3x_3$, subject to $x_1 + x_2 \geq 2$, $2x_1 + x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$.
[**Ans.** $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, min. $z = 10$]

5. Maximize $z = 2x_1 + x_2 + 3x_3$, subject to $x_1 + x_2 + 2x_3 \leq 5$, $2x_1 + 3x_2 + 4x_3 = 12$, and $x_1, x_2, x_3 \geq 0$.
What is the maximum number of basic solutions to the L.P. problem? [**Ans.** $x_1 = 3$, $x_2 = 2$, $x_3 = 0$, max. $z = 8$]

6. Maximize $z = 3x_1 + 2{\cdot}5x_2$, subject to the constraints : $2x_1 + 4x_2 \geq 40$, $3x_1 + 2x_2 \geq 50$ $x_1, x_2 \geq 0$
[**Hint.** First constraint can be divided by the common factor 2]
[**Ans.** Unbounded solution]

7. Min. $z = 0{\cdot}6x_1 + x_2$, subject to the constraints : $10x_1 + 4x_2 \geq 20$, $5x_1 + 5x_2 \geq 20$, and $x_1, x_2 \geq 0$.
[**Hint.** First divide the constraints by the common factors 2,5,2 respectively]
[**Ans.** $x_1 = 4$, $x_2 = 0$, min. $z = 12/5$]

8. Min. (cost) $z = 2y_1 + 3y_2$, subject to the constraints : $y_1 + y_2 \geq 5$, $y_1 + 2y_2 \geq 6$, $y_1 \geq 0$, $y_2 \geq 0$
[**Ans.** $y_1 = 4$, $y_2 = 1$ min. $z = 11$]

9. Max. $z = 3x_1 + 2x_2$, subject to the constraints : $2x_1 + x_2 \leq 2$, $3x_1 + 4x_2 \geq 12$, and $x_1, x_2 \geq 0$.
[**Ans.** Pseudo-optimum basic feasible solution exist.]

10. Min. $z = 4x_1 + 2x_2$, subject to the constraints : $3x_1 + x_2 \geq 27$, $x_1 + x_2 \geq 21$, and $x_1, x_2 \geq 0$.
[**Ans.** $x_1 = 3$, $x_2 = 18$, min $z = 48$]

11. Max. $z = 2x_1 + 3x_2 - 5x_3$, subject to the constraints : $x_1 + x_2 + x_3 = 7$, $2x_1 + 5x_2 + x_3 \geq 10$, and $x_1, x_2, x_3 \geq 0$.
[**Hint.** Since first constraint is an equation, we can subtract this equation from second constraint n order to reduce the number of artificial variables]
[**Ans.** $x_1 = 45/7$, $x_2 = 4/7$, $x_3 = 0$, max. $z = 102/7$]

12. Min. $z = 3x_1 + 2x_2 + x_3$, subject to : $2x_1 + 5x_2 + x_3 = 12$, $3x_1 + 4x_2 = 11$ x_1 is unrestricted, $x_2 \geq 0$, $x_3 \geq 0$.
[**Ans.** $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$, max $z = 47/3$.]

13. A cabinet manufacturer produces wood cabinets for T.V., sets, stereo systems and radios, each of which must be assembled and crated. Each T.V. cabinet requires 3 hrs. to assemble, 5 hrs. to decorate and 1/10 hr. to crate and returns a profit of Rs. 10. Each stereo cabinet requires 10 hrs. to assemble 8 hrs. to decorate and 3/5 hr. to crate and returns a profit Rs. 25. Each radio cabinet requires 1 hr. to assemble, 1 hr. to decorate and 1/10 hr. to, crate and returns a profit of Rs. 3. The manufacturer has the maximum of 30,000, 40,000 and 120 hrs. available for assembling, decorating and crating respectively.
(i) Formulate the above problem as a LPP.
(ii) Use simplex method to find how many units of each product should be manufactured to maximize profit.
(iii) Does the problem have unique solution. **[Delhi (MBA) Nov. 98]**
[**Ans.** (i) Max. $z = 10x_1 + 25x_2 + 3x_3$, s.t.
$3x_1 + 10x_2 + x_3 \leq 30{,}000$, $5x_1 + 8x_2 + x_3 \leq 40{,}000$,
$\frac{1}{10}x_1 + \frac{3}{5}x_2 + \frac{1}{10}x_3 \leq 120$, $x_1, x_2, x_3 \geq 0$
(ii) $x_1 = 1{,}200$, $x_2 = x_3 = 0$ with max. profit $z = 12{,}000$.
(iii) No.]

14. Product *A* offers a profit of Rs. 25 per unit and product *B* yields a profit of Rs. 40 per unit. To manufacture the products-leather, wood and glue are required in the amount shown below :

Resources Required for one unit

Product	Leather	Woods (in sq. units)	Glue (in litres)
A	0·50	4	0·2
B	0·25	7	0·2

Available resources include 2,200 kgs of leather, 28,000 square metres of wood and 1,400 litres of glue :
(i) State the objective function and constraints in mathematical form.
(ii) Find the optimum situation.
(iii) Which resources are fully consumed ? How much of each resource remains unused ?
(iv) What are the shadow prices of resources ? **[C.S. (Final) June 97]**
[**Hint.**(i) Max. $z = 25x_1 + 40x_2$, s.t.
$0{\cdot}50x_1 + 0{\cdot}25x_2 \geq 2{,}200$, $4x_1 + 7x_2 \leq 28{,}000$,
$0{\cdot}20x_1 + 0{\cdot}20x_2 \leq 1{,}400$, $x_1, x_2 \geq 0$.
(ii) $x_1 = 3{,}360$, $x_2 = 2{,}080$ with max. $z =$ Rs. 1,67,200

(iii) used and unused reosurces.

Resources	Used	Unused	Total
Leather	(0·50) (3360) + (0·25) (2080) = 2200 kg.	0 kg.	2,200 kg.
Wood	28,000 sq. mt.	0 sq. mt.	28,000 sq. mt.
Glue	1080 litres	312 litres	1,400 litres

15. Two products A and B are processed on 3 machine, M_1, M_2 and M_3. The processing times per unit, machine availability and profit per unit are

Machine	Processing	Time (hrs.)	Availability (hrs.)
M_1	2	3	1,500
M_2	3	2	1,500
M_3	1	1	100
Profit (Rs.)	10	12	

Any unutilized time on machine M_3 can be given on rental basis to others at an hourly rated of Rs. 1·50. Solve the problem by simplex method to determine the maximum profit. **[IGONU (MBA) Dec. 98]**

[**Hint.** Since any amount of unused time on M_3 can be rented out at a rate of Rs. 1·50 per hour, the total rent will be $1{\cdot}5\,[1000-(x_1+x_2)]$. Thus the total profit is equal to $10x_1 + 12x_2 + 1{,}500 - 1{\cdot}5x_1 - 1{\cdot}5x_2$. Thus formulation of LPP is :
Max. $z = 8{\cdot}5x_1 + 10{\cdot}5x_2 + 1{,}500$, s.t. $2x_1 + 3x_2 \le 1{,}500$, $3x_1 + 2x_2 \le 1{,}500$, $x_1 + x_2 \le 1{,}000$; $x_1, x_2 \ge 0$.
[**Ans.** 300 units of A and B both with max. profit of Rs. 7,200]

Problem of Degeneracy : Tie for Minimum Ratio

5.15 WHAT IS DEGENERACY PROBLEM ?

At the stage of improving the solution during simplex procedure, minimum ratio $\mathbf{X_B}/\mathbf{X_k}$ ($\mathbf{X_k} > \mathbf{0}$) is determined in the last column of simplex table to find the key row (*i.e.*, a row containing the *key element*). But, sometimes this ratio may not be unique, *i.e.*, the key element (hence the variable to leave the basis) is not uniquely determined or at the very first iteration, the value of one or more basic variables in the $\mathbf{X_B}$ column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon very rarely occurs in practical problems.

5.15-1 Method to Resolve Degeneracy (Tie)

The following systematic procedure can be utilised to avoid cycling due to degeneracy in L.P. problems.

Step 1. First pick up the rows for which the min. non-negative ratio is same (tied). To be definite, suppose such rows are first, third, etc., for example.

Step 2. Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.

Step 3. Then find the minimum of the ratio :

$$\left[\frac{\textbf{elements of first column of \textit{unit matrix}}}{\textbf{corresponding elements of \textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique. That is, for the rows *first, third,* etc. as picked up in step 1. (*key column* is that one for which Δ_j is minimum).

(i) If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4. Now compute the minimum of the ratio :

$$\left[\frac{\textbf{elements of second column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 3.

(i) If this min. ratio is unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5. Next compute the *minimum* of the ratio :

$$\left[\frac{\textbf{elements of third column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 4.

(i) If this min. ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this min. is still not unique, then go on repeating the above outlined procedure till the unique min. ratio is obtained to resolve the degeneracy. After the resolution of this tie, simplex method is applied to obtain the optimum solution. Following example will make the procedure clear.

Q. 1. What do you understand by degeneracy ? Discuss a method to resolve degeneracy in a LPP.

2. Explain the concept of degeneracy in simplex method. [VTU (BE Mech.) 2003]

Example 19. *Maximize* **z** $= 3x_1 + 9x_2$, *subject to the constraints* :

$$x_1 + 4x_2 \le 8,\ x_1 + 2x_2 \le 4, \text{ and } x_1, x_2 \ge 0.$$

Solution. Introducing the slack variables $s_1 \ge 0$ and $s_2 \ge 0$, the problem becomes :

$$\text{Max. } z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

subject to the constraints :

$$x_1 + 4x_2 + s_1 \quad = 8$$
$$x_1 + 2x_2 \quad + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Table 5.32. Starting Simplex Table

		$c_j \rightarrow$	3	9	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1	S_2	MIN. RATIO (X_B/X_k)
s_1	0	8	1	4	1	0	$\left\{\begin{matrix}8/4=2\\4/2=2\end{matrix}\right\}$ Tie
s_2	0	4	1	2	0	1	
	$z=0$		-3	-9	0	0	$\leftarrow \Delta_j$

Since min. ratio 2 in the last column of above table is not unique, both the slack variables s_1 and s_2 may leave the basis. This is an indication for the existence of degeneracy in the given LP problem. So we apply the above outlined procedure to resolve degeneracy (tie).

First arrange the columns $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{S}_1$ and $\mathbf{S}_2$ in such a way that the initial identity (basis) matrix appears first. Thus the initial simplex table becomes :

Table 5.33

		$c_j \rightarrow$	0	0	3	9	
BASIC VARIABLES	C_B	X_B	S_1	S_2	X_1	X_2	MIN RATIO (S_1/X_2)
s_1	0	8	1	0	1	4	1/4
$\leftarrow s_2$	0	4	0	1	1	$\leftarrow$ [2]	$0/2 \leftarrow$
	$z=0$		0	0 ↓	-3	-9 ↑	$\leftarrow \Delta_j \ge 0$

Now using the *step 3* of the procedure for resolving degeneracy, we find

$$\min\left[\frac{\textbf{elements of first column } (S_1)}{\textbf{corres. elements of } \textbf{\textit{key column}} (X_2)}\right] = \min\left[\frac{1}{4}, \frac{0}{2}\right] = 0$$

which occurs for the second row. Hence s_2 must leave the basis, and the key element is 2 as shown above.

First Iteration. By usual matrix transformation introduce x_2 and leave s_2.

Table 5.34. First Improvement Table

BASIC VARIABLES	C_B	X_B ($C_j \rightarrow$)	S_1 (0)	S_2 (0)	X_1 (3)	X_2 (9)	MIN RATIO
s_1	0	0	1	−2	−1	0	
$\rightarrow x_2$	9	2	0	1/2	1/2	1	
	$z = 18$		0	9/2	3/2	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal solution has been reached. Hence the optimum basic feasible solution is : $x_1 = 0, x_2 = 2$, max. $z = 18$.

Example 20. *Max.* $z = 2x_1 + x_2$, *subject to* $4x_1 + 3x_2 \leq 12$, $4x_1 + x_2 \leq 8$, $4x_1 - x_2 \leq 8$, *and* $x_1, x_2 \geq 0$.

Solution. Introducing the slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$, and proceeding in the usual manner, the starting simplex table is given below :

Table 5.35

BASIC VARIABLES	C_B	X_B ($c_j \rightarrow$)	X_1 (2)	X_2 (1)	S_1 (0)	S_2 (0)	S_3 (0)	MIN. RATIO (X_B/X_k)
s_1	0	12	4	3	1	0	0	12/4
s_2	0	8	4	1	0	1	0	{8/4
s_3	0	8	4	−1	0	0	1	8/4}
	$z = 0$		−2	−1	0	0	0	$\leftarrow \Delta_j$

Since min. ratio in the last column of above table is 2 which is same for *second* and *third* rows. This is an indication of *degeneracy*. So arrange the columns in such a way that the initial identity (basis) matrix comes first. Then starting simplex table becomes:

Table 5.36

BASIC VARIABLES	C_B	X_B	S_1	S_2	S_3	X_1	X_2	MIN (S_1/X_1)	MIN (S_2/X_1)
s_1	0	12	1	0	0	4	3	—	—
s_2	0	8	0	1	0	4	1	0/4	1/4
s_3	0	8	0	0	1	←[4]	1	0/4	0/4←
	$z = 0$		0	0	0 ↓	−2 ↑	−1	$\leftarrow \Delta_j$	

Using the procedure of degeneracy, compute

$$\left[\frac{\textbf{elements of first column } (S_1) \textbf{ of unit matrix}}{\textbf{corres. elements of key column } (X_1)}\right]$$

only for *second* and *third* rows. Therefore, min $[-, \frac{0}{4}, \frac{0}{4}]$ which is not unique.

So again compute

$$\min\left[\frac{\textbf{element of second column } (S_2) \textbf{ of unit matrix}}{\textbf{corres. element of key column } (X_1)}\right]$$

only for second and third rows. Therefore, min $[-, \frac{1}{4}, \frac{0}{4}] = 0$ which occurs corresponding to the third row. Hence the key element is 4.

Now improve the simplex **Table 5.36** in the usual manner to get Table 5.37.

Table 5.37

BASIC VARIABLES	C_B	X_B	S_1 (0)	S_2 (0)	S_3 (0)	X_1 (2)	X_2 (1)	MIN. (X_B/X_k)
s_1	0	4	1	0	−1	0	4	4/4
s_2	0	0	0	1	−1	0	[2]	0/2 ←
x_1	2	2	0	0	1/4	1	−1/4	—
	$z=4$		0	0 ↓	1/2	0	−3/2 ↑	← Δ_j
s_1	0	4	1	−2	[1]	0	0	4/1←
x_2	1	0	0	1/2	−1/2	0	1	—
x_1	2	2	0	1/8	1/8	1	0	$2/\frac{1}{8}$
	$z=4$		0 ↓	3/4	−1/4 ↑	0	0	← Δ_j
s_3	0	4	1	−2	1	0	0	
x_2	1	2	1/2	−1/2	0	0	1	
x_1	2	3/2	−1/8	3/8	0	1	0	
	$z=5$		1/4	1/4	0	0	0	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum solution is obtained as : $x_1 = 3/2, x_2 = 2$, max $z = 5$.

Example 21. *Max.* $z = 5x_1 - 2x_2 + 3x_3$, *subject to* $2x_1 + 2x_2 - x_3 \geq 2$, $3x_1 - 4x_2 \leq 3$, $x_2 + 3x_3 \leq 5$, *and* $x_1, x_2, x_3 \geq 0$. **[Kanpur 96]**

Solution. Introducing the surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0$, $s_3 \geq 0$ and an artificial variable $a_1 \geq 0$, the constraints of the problem become :

$$2x_1 + 2x_2 - x_3 - s_1 + a_1 = 2$$
$$3x_1 - 4x_2 + s_2 = 3$$
$$x_2 + 3x_3 + s_3 = 5.$$

and using big-M technique objective function becomes :

$$\text{Max. } z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1.$$

In the usual manner, the starting simplex table is obtained as below :

Table 5.38

BASIC VARIABLES	C_B	X_B	X_1 (5)	X_2 (−2)	X_3 (3)	S_1 (0)	S_2 (0)	S_3 (0)	A_1 (−M)	MIN. RATIO (X_B/X_k)
← a_1	−M	2	[2]	2	−1	−1	0	0	1	2/2←
s_2	0	3	3	−4	0	0	1	0	0	3/3
s_3	0	5	0	1	3	0	0	1	0	--
	$z=-2M$		−2M−5 ↑	−2M+2	M−3	M	0	0	0 ↓	← Δ_j

Net evaluations Δ_j are computed by the formula $\Delta_j = c_B X_j - c_j$ in the usual manner. Since Δ_1 is the most negative, $\mathbf{X}_1$ enters the basis. Further, since the min. ratio in the last column of above table is 1 for both the first and second rows, therefore either $\mathbf{A}_1$ or $\mathbf{S}_2$ tends to leave the basis. This is an indication of the existence of degeneracy. But, $\mathbf{A}_1$ being an artificial vector will be prefered to leave the basis. *Note that there is no need to apply the procedure for resolving degeneracy under such circumstances.*

Continuing the simplex routine, the computations are presented in the following tabular form.

Table 5.39

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	MIN. RATIO (X_B/X_k)
		$c_j \rightarrow$	5	−2	3	0	0	0	
$\rightarrow x_1$	5	1	1	1	−1/2	−1/2	0	0	–
$\leftarrow s_2$	0	0	0	−7	[3/2]	3/2	−1	0	$0/\frac{3}{2} \leftarrow$
s_3	0	5	0	1	3	0	0	1	5/3
	$z = 5$		0	7	−11/2 ↑	−5/2	0 ↓	0	$\leftarrow \Delta_j$
x_1	5	1	1	−4/3	0	0	1/3	0	–
$\rightarrow x_3$	3	0	0	−14/3	1	1	2/3	0	–
$\leftarrow s_3$	0	5	0	[15]	0	−3	−2	1	5/15 ←
	$z = 5$		0	−56/3 ↑	0	3	11/3	0 ↓	$\leftarrow \Delta_j$
x_1	5	13/9	1	0	0	−4/15	7/45	4/45	–
$\leftarrow x_3$	3	14/9	0	0	1	[1/15]	2/45	14/45	70/3 ←
$\rightarrow x_2$	−2	1/3	0	1	0	−1/5	−2/15	1/15	–
	$z = 101/9$		0	0	0 ↓	−11/15 ↑	53/45	56/45	$\leftarrow \Delta_j$
x_1	5	23/3	1	0	4	0	1/3	4/3	
$\rightarrow s_1$	0	70/3	0	0	15	1	2/3	14/3	
x_2	−2	5	0	1	3	0	0	1	
	$z = 85/3$		0	0	11	0	5/3	14/3	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is : $x_1 = 23/3, x_2 = 5, x_3 = 0$, max. $z = 85/3$.

Q. 1. What is degeneracy ? Discuss a method to resolve degeneracy in L.P. problems.

2. Explain what is meant by degeneracy and cycling in linear programming. How their effects overcome ?

[Meerut (L.P.) 90]

EXAMINATION PROBLEMS

Solve the following LP problems :

1. Max. $z = 5x_1 + 3x_2$
   subject to
   $x_1 + x_2 \leq 2$
   $5x_1 + 2x_2 \leq 10$
   $3x_1 + 8x_2 \leq 12$
   $x_1, x_2, \geq 0.$
   [Ans. $x_1 = 2, x_2 = 0, z = 10$]

2. Max. $R = 22x + 30y + 25z$
   subject to
   $2x + 2y \leq 100$
   $2x + y + z \leq 100$
   $x + 2y + 2z \leq 100$
   $x, y, z, \geq 0$.
   [Ans. $x = 100/3, y = 50/3, z = 50/3,$] $R = 1650$

3. Max. $z = 2x_1 + 3x_2 + 10x_3$
   subject to
   $x_1 + 2x_3 = 0$
   $x_2 + x_3 = 1$
   $x_1, x_2, x_3 \geq 0$.
   [Ans. $x_1 = 0, x_2 = 1, x_3 = 0$ and max. $z = 3$]

4. Max. $z = 3x_1 + 5x_2$
   subject to the constraints
   $x_1 + x_3 = 4, x_2 + x_4 = 6,$
   $3x_1 + 2x_2 + x_5 = 12$, and
   $x_1, x_2, x_3, x_4, x_5 \geq 0$
   Does the degeneracy occur in this problem ?
   [Ans. $x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 0,$ $z^* = 30$. Yes, degeneracy occurs.]

5. Max. $z = 2x_1 + x_2$
   subject to the constraints
   $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6,$
   $x_1 - x_2 \leq 2, x_1 + 2x_2 \leq 1,$
   $2x_1 - 3x_2 \leq 1$, and $x_1, x_2 \geq 0$.
   [Ans. $x_1 = 5/7, x_2 = 1/7$ max. $z = 11/7$]

6. Max. $z = 3/4\, x_1 - 150\, x_2 + 1/50\, x_3 - 6x_4,$
   subject to the constraints
   $1/4\, x_1 - 60\, x_2 - 1/26\, x_3 + 9\, x_4 \leq 0,$
   $1/2\, x_1 - 90\, x_2 - 1/50\, x_3 + 3x_4 \leq 0,$
   $x_3 \leq 1$ and $x_1, x_2, x_3, x_4 \geq 0.$
   [Ans. $x_1 = 1/25, x_2 = 0, x_3 = 1$ and $x_4 = 0$, max. $z = 1/20$]

7. Max. $z = 2x_1 + 3x_2 + 10x_3$, subject to
   $x_1 + 2x_3 = 1, x_2 + x_3 = 1,$
   and $x_1, x_2, x_3 \geq 0.$
   [Ans. $x_1 = 0, x_2 = 1/2, x_3 = 1/2$, max. $z = 13/2$]

8. Min. $z = -3/4\, x_1 + 20x_2 - 1/2\, x_3 + 6x_4$, subject to
   $1/4\, x_1 - 8x_2 - x_3 + 9x_4 \leq 0,\ 1/4\, x_1 - 12x_2 - 1/2\, x_3 + 3x_4 \leq 0$
   and $x_1, x_2, x_3, x_4 \geq 0.$
   [Ans. Unbounded solution.]

5·16 SPECIAL CASES : ALTERNATIVE UNBOUNDED AND NON-EXISTING SOLUTIONS

In this section, some important cases (except degeneracy) are discussed which are very often encountered during simplex procedure. The properties of these cases have already been visualised in the graphical solution of two variable LP problems.

5·16-1 Alternative Optimum Solutions

Example 22. *Use penalty (or Big-M) method to solve the problem :*

Max. $z = 6x_1 + 4x_2$, *subject to* $2x_1 + 3x_2 \leq 30$, $3x_1 + 2x_2 \leq 24$, $x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$.

Is the solution unique ? If not, give two different solutions.

Solution. Introducing the slack variables $x_3 \geq 0$, $x_4 \geq 0$, surplus variable $x_5 \geq 0$, and artificial variable $a_1 \geq 0$, the problem becomes :

Max. $z = 6x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$2x_1 + 3x_2 + x_3 = 30$$
$$3x_1 + 2x_2 + x_4 = 24$$
$$x_1 + x_2 - x_5 + a_1 = 3$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0.$$

Now the solution is obtained as follows :

Table 5·40

	$c_j \rightarrow$		6	4	0	0	0	−M	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN RATIO (X_B/X_k)
x_3	0	30	2	3	1	0	0	0	30/2
x_4	0	24	3	2	0	1	0	0	24/3
$\leftarrow a_1$	$-M$	3	[1]	1	0	0	−1	1	3/1 ←
	$z = -3M$		$(-M-6)$ ↑	$(-M-4)$	0	0	M	0 ↓	$\leftarrow \Delta_j$
x_3	0	24	0	1	1	0	2	×	24/2
$\leftarrow x_4$	0	15	0	−1	0	1	[3]	×	15/3 ←
$\rightarrow x_1$	6	3	1	1	0	0	−1	×	—
	$z = 18$		0	2	0	0 ↓	−6 ↑	×	$\leftarrow \Delta_j$
$\leftarrow x_3$	0	14	0	[5/3]	1	−2/3	0	×	$\frac{14}{5/3} = 42/5$ ←
$\rightarrow x_5$	0	5	0	−1/3	0	1/3	1	×	—
x_1	6	8	1	2/3	0	1/3	0	×	$\frac{8}{2/3} = 12$
	$z = 48$		0	0* ↑	0 ↓	2	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is obtained as : $x_1 = 8$, $x_2 = 0$, max $z = 48$.

Alternative Solutions. Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that the *alternative solutions* also exist. Therefore, the solution as obtained above is not unique.

Thus we can bring $\mathbf{X_2}$ into the basis in place of $\mathbf{X_3}$. Therefore, introducing $\mathbf{X_2}$ into the basis in place of $\mathbf{X_3}$, the new optimum simplex table is obtained as follows :

Table 5·41

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_2	4	42/5	0	1	3/5	−2/5	0	×	
x_5	0	39/5	0	0	1/5	1/5	1	×	
x_1	6	12/5	1	0	−2/5	3/5	0	×	
		$z=48$	0	0	0	2	0	×	$\leftarrow \Delta_j \geq 0$

From this table we get a different optimum solution : $x_1 = 12/5$, $x_2 = 42/5$, max. $z = 48$.

Thus, if two alternative optimum solutions can be obtained, then any number of optimum solutions can be obtained, as given below :

Variables	First Sol.	Second. Sol.	General Solution
x_1	8	12/5	$x_1 = 8\lambda + (12/5)(1-\lambda)$
x_2	0	42/5	$x_2 = 0\lambda + (42/5)(1-\lambda)$
x_3	14	0	$x_3 = 14\lambda + 0(1-\lambda)$
x_4	0	0	$x_4 = 0\lambda + 0(1-\lambda)$
x_5	5	39/5	$x_5 = 5\lambda + (39/5)(1-\lambda)$
a_1	0	0	$a_1 = 0\lambda + 0(1-\lambda)$

For any arbitrary value of λ, same optimal value of z will be obtained.

Note. If two optimum solutions of an LP problem are obtained. thus the mean of these two solutions will give us the third optimum solution. This process can be continued indefinitely to get as many alternative solutions as we want.

Example 23. *Maximize* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to the constraints :*

$x_1 + 2x_2 + 3x_3 = 15, 2x_1 + x_2 + 5x_3 = 20, x_1 + 2x_2 + x_3 + x_4 = 10,$ *and* $x_1, x_2, x_3, x_4 \geq 0$.

Solution. Introducing artificial variables a_1 and a_2 in the *first* and *second* constraint equations, respectively, and the *original variable* x_4 can be treated to work as an artificial variable for the *third* constraint equation to obtain :

$$x_1 + 2x_2 + 3x_3 + a_1 = 15$$
$$2x_1 + x_2 + 5x_3 + a_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10 .$$

Phase 1 : Table 5·42

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	15	1	2	3	0	1	0
a_2	20	2	1	5	0	0	1
$\leftarrow x_4$	10	[1] ↑	2	1	1 ↓	0	0

By the same arguments as given in the previous examples of two-phase method insert $\mathbf{X}_4$ in place of $\mathbf{X}_1$. The transformed table (*Table 5·43*) is obtained by applying row transformations $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$.

Table 5·43

BASIC VARIABES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	5	0	0	2	−1	1	0
$\leftarrow a_2$	0	0	−3	[3]	−2	0	1
$\rightarrow x_1$	10	1	2	1 ↑	1	0	0 ↓

In spite of the fact that the artificial variable x_4 has served its purpose, the column $\mathbf{X}_4$ cannot be deleted from *Table 5·43*, because x_4 is the original variable also. Although the value of the artificial variable a_2 also becomes zero at this stage, the column $\mathbf{A}_2$ cannot be deleted unless it is inserted at one of the places $\mathbf{X}_2$ or $\mathbf{X}_3$ or

$\mathbf{X}_4$ (wherever it is possible). Now, it is observed that $\mathbf{A}_2$ can be inserted in place of $\mathbf{X}_3$. Hence transformation *Table* 5·44 is obtained by applying the row transformations : $R_2 \to \frac{1}{3} R_2$, $R_1 \to R_1 - \frac{2}{3} R_2$, $R_3 \to R_3 - \frac{1}{3} R_2$.

Table 5·44

BASIC VARIABLES	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
$\leftarrow a_1$	5	0	2	0	1/3	1	-2/3
$\to x_3$	0	0	-1	1	-2/3	0	1/3
x_1	10	1	3	0	5/3	0	-4/3
			↑			↓	

Now removing $\mathbf{A}_1$ and inserting it in the suitable position of $\mathbf{X}_2$, the next transformed *Table 5·45* is obtained by row transformations : $R_1 \to \frac{1}{2} R_1$, $R_2 \to R_2 + \frac{1}{2} R_1$, $R_3 \to R_3 - \frac{3}{2} R_1$.

Table 5·45

BASIC VARIABLES	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$
x_2	5/2	0	1	0	1/6	1/2
x_3	5/2	0	0	1	-1/2	1/2
x_1	5/2	1	0	0	7/6	-3/2

Delete column $\mathbf{A}_1$ ($a_1 = 0$). The starting basic feasible solution is obtained : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$.
Further, proceed to test this solution for optimality in *Phase II*. For this, compute

$$\Delta_4 = \mathbf{C}_B\mathbf{X}_4 - c_4 = (2, 3, 1)(1/6, -1/2, 7/6) - 0 = 0.$$

Phase II. Table 5·46

BASIC VARIABLES	$\mathbf{C}_B$	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	Min. Ratio
x_2	2	5/2	0	1	0	1/6	
x_3	3	5/2	0	0	1	-1/2	
x_1	1	5/2	1	0	0	7/6	
	$z = \mathbf{C}_B\mathbf{X}_B = 15$		0	0	0	0*	$\leftarrow \Delta_j$

Since all Δ_j's are zero, the solution : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, is optimal to give us $z^* = 15$. Further, Δ_4 being zero indicates that *alternative* optimal solutions are also possible.

Note. Here Δ_j corresponding to nonbasic vector $\mathbf{X}_4$ also becomes zero. This indicates that alternative optimum solutions are possible. However, the other optimal solutions can be obtained as : $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$, max. $z = 15$.

Now, given the *two* alternative basic solutions ;

(i) $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$ (ii) $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$

an infinite number of *non-basic* solutions can be obtained and by realizing them any weighted average of these two basic solutions is also an alternative optimum solution.

To verify this, *third* solution will be obtained as :

$$x_1 = \frac{5/2 + 0}{2},\ x_2 = \frac{5/2 + 15/7}{2},\ x_3 = \frac{5/2 + 25/7}{2},\ x_4 = \frac{0 + 0}{2}$$

i.e., $x_1 = 5/4$, $x_2 = 65/28$, $x_3 = 85/28$, $x_4 = 0$,

yielding the maximum value of $z = 15$.

Note. Also see example 14 page 134.

Example 24. *Following is the LP problem : Maximize* $z = x_1 + x_2 + x_4$, *subject to the constraints* :

$$x_1 + x_2 + x_3 + x_4 = 4,\ x_1 + 2x_2 + x_3 + x_5 = 4,\ x_1 + 2x_2 + x_3 = 4,\ x_1, x_2, x_3, x_4, x_5 \ge 0.$$

(i) *Find out all the optimal basic feasible solutions by using penalty (or Big-M) method.*

(ii) *Write-down the general form of an optimal solution.*

Solution. Since the constraints of the given problem are already equations, only artificial variables are required to form the basis matrix. In order to bring the basis matrix as unit matrix, only artificial variable $a_1 \ge 0$ is needed in the third constraint. So the problem may be re-written in the form :

Max. $z = x_1 + x_2 + 0x_3 + x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$x_1 + x_2 + x_3 + x_4 \quad = 4$$
$$x_1 + 2x_2 + x_3 \quad + x_5 \quad = 4$$
$$x_1 + 2x_2 + x_3 \quad + a_1 = 4$$
$$x_1, x_2, \ldots, x_5, a_1 \geq 0$$

These constraints may be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}.$$

Applying the usual simplex method, the solution is obtained as follows :

Table 5·47

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	4	1	1	1	1	0	0	4/1
x_5	0	4	1	2	1	0	1	0	4/2
$\leftarrow a_1$	$-M$	4	1	[2]	1	0	0	1	4/2 ←(Note)
	$z = -4M + 4$		$-M$	$-2M$ ↑	$-M+1$	0	0	0 ↓	$\leftarrow \Delta_j$

Note. Here it is observed that the minimum 4/2 occurs at two places (2nd and 3rd) in the last column. Although one of these two may be chosen by degeneracy rule (see 5.7-1, page 140), but minimum at 3rd place has been chosen to remove artificial basis vector A_1 from the basis matrix.

Table 5·48

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	2	1/2	0	1/2	1	0	×	
x_5	0	0	0	0	0	0	1	×	
$\leftarrow x_2$	1	2	1/2	1	1/2	0	0	×	
	$z = 4$		0*	0	1	0	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained. Thus the optimum solution is given by

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2, x_5 = 0, \text{ max. } z = 4.$$

Since $\Delta_1 = 0$, alternative optimum solutions also exist.

5·16–2 Unbounded Solutions

The case of unbounded solutions occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely. It is not necessary, however, that an unbounded feasible region should yield an unbounded value for the objective function. The following examples will illustrate these points.

Example 25. (Unbounded Optimal Solution)

Max. $z = 2x_1 + x_2$, *subject to* : $x_1 - x_2 \leq 10$, $2x_1 - x_2 \leq 40$, *and* $x_1 \geq 0$, $x_2 \geq 0$.

Solution. The starting simplex table is as follows :

Table 5·49

BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1 (β_1)	S_2 (β_2)
s_1	0	10	1	−1	1	0
s_2	0	40	2	−1	0	1
	$z = C_B X_B = 0$		−2	−1	0	0

It can be seen from the starting simplex table that the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are candidates for the entering vector. Since Δ_1 has the minimum value, $\mathbf{X}_1$ should be selected as the entering vector. It is noticed, however, that if $\mathbf{X}_2$ is selected as the entering vector, the value of x_2 (and hence the value of z) can be increased indefinitely without affecting the feasibility of the solution (since it has all x_{i2} negative). It is thus concluded that the problem has no bounded solution. This can also be seen from the graphical solution of the problem in Fig. 5.1.

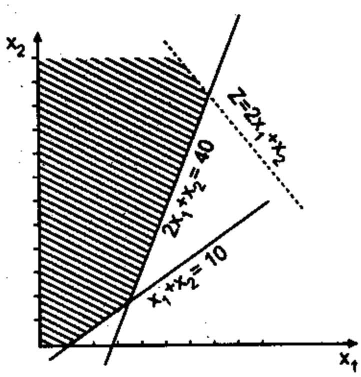


Fig. 5.1

In general, *an unbounded solution can be detected if, at any iteration, any of the candidates for the entering vector* $\mathbf{X}_k$ *(for which* $\Delta_k < 0$*, i.e.* $z_k - c_k < 0$*) has all* $x_{ik} \le 0$, $i = 1, 2, ..., m$, *i.e., all elements of the entering column are* ≤ 0.

Example 26. (Unbounded Solution)

Maximize $z = 107x_1 + x_2 + 2x_3$, *subject to* :

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7,\ 16x_1 + \tfrac{1}{2}x_2 - 6x_3 \le 5,\ 3x_1 - x_2 - x_3 \le 0,\ \text{and}\ x_1, x_2, x_3 \ge 0.$$

Solution. By introducing slack variables, $x_5 \ge 0$, $x_6 \ge 0$, the set of constraints is converted into the system of equations :

$$\begin{cases} 14x_1 + x_2 - 6x_3 + 3x_4 & = 7 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 & = 5 \\ 3x_1 - x_2 - x_3 + x_6 & = 0 \end{cases} \quad \text{or} \quad \begin{cases} \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 & = 7/3 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 & = 5 \\ 3x_1 - x_2 - x_3 + x_6 = 0 \end{cases}$$

or

$$\begin{bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

Here original variable x_4 has been treated as slack variable as its coefficient in the objective function is zero, *i.e.*, *Maximize* $z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Now start simplex method as follows :

Table 5·50

		$c_j \to$	107	1	2	0	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	MIN. RATIO
x_4	0	7/3	14/3	1/3	−2	1	0	0	7/14
x_5	0	5	16	1/2	−6	0	1	0	5/16
x_6	0	0	← [3]	−1	−1	0	0	1	0/3 ←
	$z = 0$		−107 ↑	−1	−2	0	0	0 ↓	← Δ_j
x_4	0	7/3	0	17/9	−4/9	1	0	−14/9	
x_5	0	5	0	35/6	−2/3	0	1	−16/3	
x_1	107	0	1	−1/3	−1/3	0	0	1/3	
	$z = 0$		0	$-\frac{110}{3}$	$-\frac{113}{3}$	0	0	$\frac{107}{3}$	← Δ_j

Since corresponding to negative Δ_3, all elements of $\mathbf{X}_3$ column are negative, so $\mathbf{X}_3$ cannot enter into the basis matrix. Consequently, this is an indication that there exists an *unbounded solution* to the given problem.

Example 27. ***(Unbounded feasible region but bounded optimal solution)***

Max. $z = 6x_1 - 2x_2$, *subject to* $2x_1 - x_2 \le 2, x_1 \le 4$, *and* $x_1, x_2 \ge 0$.

Solution. We only give the successive tables here. Students are advised to fill up the details.

Table 5·51. Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (c_j = 6)	X_2 (c_j = −2)	X_3 (β_1) (c_j = 0)	X_4 (β_2) (c_j = 0)	MIN. RATIO (X_B/X_1)
x_3	0	2	[2]	−1	1	0	2/2 ←
x_4	0	4	1	0	0	1	4/1
	$z = C_BX_B = 0$		−6 ↑	2	0 ↓	0	← Δ_j

First Improvement. We enter X_1 and remove β_1.

Table 5·52

BASIC VARIABLES	C_B	X_B	X_1 (β_1) (c_j = 6)	X_2 (c_j = −2)	X_3 (c_j = 0)	X_4 (β_2) (c_j = 0)	MIN. RATIO (X_B/X_2)
x_1	6	1	1	−1/2	1/2	0	—
x_4	0	3	0	[1/2]	−1/2	1	$3/\frac{1}{2}$ ←
	$z = C_BX_B = 6$		0	−1 ↑	3	0 ↓	← Δ_j

Second Improvement. Enter X_2 and remove β_2.

Table 5·53

BASIC VARABLES	C_B	X_B	X_1 (β_1)	X_2 (β_2)	X_3	X_4	Min. Ratio
x_1	6	4	1	0	0	1	
x_2	−2	6	0	1	−1	2	
	$z = C_BX_B = 12$		0	0	2	2	← Δ_j

The optimal solution is : $x_1 = 4$, $x_2 = 6$, and $z = 12$.

It is now interesting to note from starting table that the elements of X_2 are negative or zero (−1 and 0). This is an immediate indication that the feasible region is not bounded (see Fig. 5·2). From this, we conclude that a problem may have unbounded feasible region but still the optimal solution is bounded.

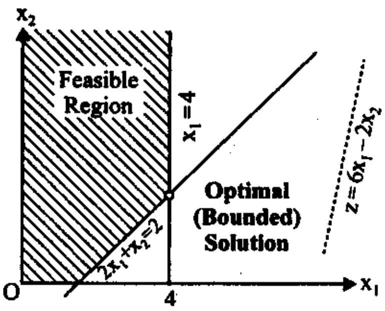


Fig. 5.2

5·16–3 Non-existing feasible solutions

In this case, the feasible region is found to be empty which indicates that the problem has no feasible solution. The following example shows how such a situation can be detected by simplex method.

Example 28. ***(Problem with no feasible solution).***

Max. $z = 3x_1 + 2x_2$, *subject to* $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, *and* $x_1, x_2 \ge 0$.

[Garhwal 97; Meerut (O.R.) 90]

Solution. Introducing slack variable x_3, surplus variable x_4 together with the artificial variable a_1, the constraints become :

$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 + a_1 = 12.$$

Here we use M-technique for dealing with artificial variable a_1. For this, we write the objective function as

$$\text{Max. } z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Ma_1 .$$

The starting simplex table will be as follows.

Table 5·54

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (β_1)	X_4	A_1 (β_2)	MIN. RATIO (X_B/X_K)
		$c_j \rightarrow$	3	2	0	0	$-M$	
$\leftarrow x_3$	0	2	1	[1]	1	0	0	2/1 ←
a_1	$-M$	12	3	4	0	−1	1	12/4
	$z = C_BX_B = -12M$		$(-3M-3)$	$(-4M-2)$ ↑	0 ↓	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_BX_1 - c_1 = (0, -M)(2, 3) - 3 = (0 - 3M) - 3 = -3 - 3M$$
$$\Delta_2 = C_BX_2 - c_2 = (0, -M)(1, 4) - 2 = (0 - 4M) - 2 = -2 - 4M$$
$$\Delta_4 = C_BX_4 - c_4 = (0, -M)(0, -1) - 0 = M .$$

First improvement. Inserting X_2 and removing β_1, *i.e.* X_3

Table 5·55

BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1	S_2	A_1	MIN. RATIO
		$c_j \rightarrow$	3	2	0	0	$-M$	
x_2	2	2	2	1	1	0	0	
a_1	$-M$	4	−5	0	−4	−1	1	
	$z = C_BX_B = 4 - 4M$		$(1+5M)$	0	$(2+4M)$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_BY_1 - c_1 = (2, -M)(2, -5) - 3 = (4 + 5M) - 3 = (1 + 5M)$$
$$\Delta_3 = C_BY_3 - c_3 = (2, -M)(1, -4) - 0 = (2 + 4M) - 0 = (2 + 4M)$$
$$\Delta_4 = C_BY_4 - c_4 = (2, -M)(0, -1) - 0 = (0 + M) = M .$$

Here all Δ_j are positive since $M > 0$. So according to the optimality condition, this solution is optimal.

Note. Here we should, however, note that the optimal (basic) solution :

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 0, a_1 = 4,$$

includes the artificial variable a_1 with positive value 4. This immediately indicates that the problem has no feasible solution, because the positive value of a_1 violates the second constraint of given problem. This situation can be observed by the graphical representation of this example in Fig. 5·3.

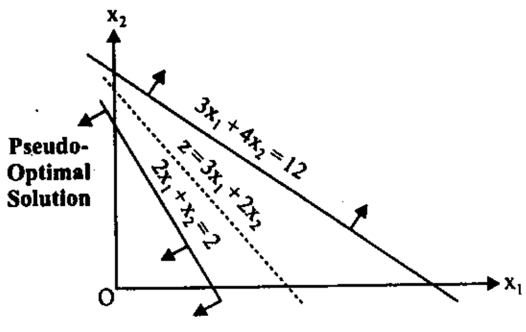


Fig. 5.3.

Such solution may be called "*pseudo-optimal*", since (as clear from the Figure 5·3) it does not satisfy all the constraints, but it satisfies the optimality condition of the simplex method. **[JNTU (B.Tech) 98]**

5·17 SOLUTION OF SIMULTANEOUS EQUATIONS BY SIMPLEX METHOD

For the solution of n simultaneous linear equations in n variables a *dummy* objective function is introduced as

$$\text{Max. } z = 0\,X - 1\,X_a$$

where X_a are artificial variables, and $x_r = x_r' - x_r''$, such that $x_r' \geq 0$, $x_r'' \geq 0$.

The reformulated linear programming problem is then solved by simplex method. The optimal solution of this problem gives the values of the variables (X).

The following example will illustrate the procedure.

Example 29. *Use simplex method to solve the following system of linear equations* :

$$x_1 - x_3 + 4x_4 = 3, \; 2x_1 - x_2 = 3, \; 3x_1 - 2x_2 - x_4 = 1, \text{ and } x_1, x_2, x_3, x_4 \geq 0 .$$

[Meerut M. Com. Jan. 98 (BP)]

Solution. Since the objective function for the given constraint equation is not prescribed, so a dummy objective function is introduced as :

Max. $z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1a_1 - 1a_2 - 1a_3$, where $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$ are artificial variables.
Introducing artificial variables, the given equations can be written as :

$$\begin{aligned} x_1 \quad - x_3 + 4x_4 + a_1 \quad &= 3 \\ 2x_1 - x_2 \qquad + a_2 \quad &= 3 \\ 3x_1 - 2x_2 - x_4 \qquad + a_3 &= 1 . \end{aligned}$$

Now apply simplex method to solve the reformulated problem as shown in *Table 5·56*.

Table 5·56

		$c_j \rightarrow$	0	0	0	0	−1	−1	−1	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	A_3	MIN RATIO (X_B/X_k)
a_1	−1	3	1	0	−1	4	1	0	0	3/1
a_2	−1	3	2	−1	0	0	0	1	0	3/2
$\leftarrow a_3$	−1	1	[3]	−2	0	−1	0	0	1	1/3 ←
	$z = -10$		−6 ↑	3	1	−3	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	−1	8/3	0	2/3	−1	[13/3]	−1	0	×	8/13 ←
a_2	−1	7/3	0	1/3	0	2/3	0	1	×	7/2
$\rightarrow x_1$	0	1/3	1	−2/3	0	−1/3	0	0	×	—
	$z = -5$		0	−1	1	−5 ↑	0 ↓	0	×	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	8/13	0	[2/13]	3/13	1	×	0	×	8/2 ←
a_2	−1	25/13	0	3/13	2/13	0	×	1	×	25/3
x_1	0	7/13	1	−8/13	1/13	0	×	0	×	—
	$z = 25/13$		0	−3/13 ↑	−2/13	0 ↓	×	0	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	0	4	0	1	−3/2	13/2	×	0	×	—
$\leftarrow a_2$	−1	1	0	0	[1/2]	−3/2	×	1	×	$1/\frac{1}{2}$ ←
x_1	0	3	1	0	−1	4	×	0	×	—
	$z = -1$		0	0	−1/2 ↑	3/2	×	0 ↓	×	$\leftarrow \Delta_j$
x_2	0	7	0	1	0	2	×	×	×	
$\rightarrow x_3$	0	2	0	0	1	−3	×	×	×	
x_1	0	5	1	0	0	1	×	×	×	
	$z = 0$		0	0	0	0	×	×	×	$\leftarrow \Delta_j = 0$

Since all $\Delta_j = 0$, an optimum solution has been attained. Thus the solution of simultaneous equations is given by, $x_1 = 5$, $x_2 = 7$, $x_3 = 2$, and $x_4 = 0$.

5·18 INVERSE OF A MATRIX BY SIMPLEX METHOD

Let **A** be any $n \times n$ real matrix. Let $\mathbf{X} \in R^n$ and **b** be any dummy $n \times 1$ real matrix. Then, consider the system of equations : $\mathbf{AX} = \mathbf{b}$; $\mathbf{X} \geq \mathbf{0}$.

By introducing a *dummy* objective function $z = 0\,\mathbf{X} - 1\mathbf{X}_a$, where $\mathbf{X}_a \geq 0$ being artificial variable vector, then we find a solution to the LPP of maximizing z subject to the constraints : $\mathbf{AX} + 1\,\mathbf{X}_a = \mathbf{b}$; $\mathbf{X}, \mathbf{X}_a \geq \mathbf{0}$.

If we get the optimal solution to the given LPP in which the basis contains all the variables of vector **X**, then inverse of **A** is directly read off from the optimum (final) simplex table. Then inverse of **A** consists of those column vectors in the last iteration of the simplex method which were present in the initial basis **B**. In addition, if in the last iteration the columns of **A** become the columns of **I**, then $\mathbf{B}^{-1} = \mathbf{A}^{-1}$.

If the optimum (final) simplex table does not contain all the variables of vector **X** in the basis, we continue simplex procedure until all the variables of vector **X** are in the basis and at the same time the solution remains optimum, may be feasible or infeasible.

Note. The dummy vector **b** can be constructed easily by assigning the value one to all the variables of vector **X**.

Following example will illustrate the procedure :

Example 30. *Use simplex method to obtain the inverse of the matrix* $A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$.

Solution. Consider the matrix equation

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; x_1, x_2 \geq 0,$$

where the right side $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is a *dummy vector.* Now apply simplex method to maximize the dummy objective function $z = 0x_1 + 0x_2 - 1x_{3a} - 1x_{4a}$, subject to the above constraints on x_1, x_2, where $x_{3a} \geq 0$ and $x_{4a} \geq 0$ are artificial variables.

Table 5·57

BASIC VARIABLES	C_B	X_B	X_1 (0)	X_2 (0)	X_{3A} (−1)	X_{4A} (−1)	MIN. RATIO (X_B/X_K)
x_{3a}	−1	4	[3]	2	1	0	4/3 ←
x_{4a}	−1	6	4	−1	0	1	6/4
	$z = -10$		−7 ↑	−1	0 ↓	0	← Δ_j
→ x_1	0	4/3	1	2/3	1/3	0	
← x_{4a}	−1	2/3	0	[−11/3]	−4/3	1	
	$z = -2/3$		0	11/3 ↑	7/3	0 ↓	← $\Delta_j \geq 0$
x_1	0	16/11	1	0	1/11	2/11	
x_2	0	−2/11	0	1	4/11	−3/11	
	$z = 0$		0	0	1	1	← Δ_j

Since all $\Delta_j \geq 0$, an optimum solution is obtained. But matrix A is not yet converted into unit matrix. To do so, introduce X_2 in the basis and remove X_{4A} from the basis.

Now an optimum (but infeasible) solution has been obtained. Since the initial basis consisted of column vectors X_{3A} and X_{4A}, the inverse of matrix A is given by

$$\begin{bmatrix} 1/11 & 2/11 \\ 4/11 & -3/11 \end{bmatrix}.$$

Note. Here *key element* may be *negative* also.

Example 31. *Following is the final optimal table for a given L.P. problem, answer that it originally has an identity matrix under* X_3 and X_4.

(a) *What is the value of the objective function for optimal solution ?*

(b) *What is the optimal basis ? Give a* 2 × 2 *numerical matrix.*

(c) *Are there any alternative optimal solutions ? If so, which variable gives an alternative optimal solution ?*

Table 5·58

BASIC VARIABLES	C_B	X_B	X_1 (2)	X_2 (2)	X_3 (0)	X_4 (0)
x_1	2	5	1	0	1/2	−1/2
x_2	2	4	0	1	−1/2	3/2
			0	0	0	2

(d) *Suppose* c_2 *was equal to 3 instead of 2. Would we have still the optimal solution.*

(e) *Without calculating write the inverse of the optimal basis found in (b) above.*

Solution. (a) The optimal solution is given as $x_1 = 5$, $x_2 = 4$.

The value of objective function will be

$$z = C_B X_B = (2, 2)(5, 4) = 2 \times 5 + 2 \times 4 = 18.$$

(b) Find the optimal basis **B** as follows :

Since $\mathbf{B}^{-1}=\begin{bmatrix}1/2 & -1/2\\ -1/2 & 3/2\end{bmatrix}$, and $\mathbf{BB}^{-1}=\mathbf{I}$, we have $\mathbf{B}\begin{bmatrix}1/2 & -1/2\\ -1/2 & 3/2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Now apply the elementary row operations : $R_1 \to 2R_1$ and $R_2 \to 2R_2$ and we get

$$\mathbf{B}\cdot\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

Again applying $R_2 \to R_2 + R_1$, $\mathbf{B}.\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}$

Further applying $R_2 \to (1/2)\,R_2$, $\mathbf{B}.\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$

Again applying $R_1 \to R_1 + R_2$, $\mathbf{B}.\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}$ or $\mathbf{BI}=\mathbf{B}=\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}$. **Ans.**

(c) Since $\Delta_3 = 0$ and the variable x_3 is not in the basis, it shows that alternative optimum solutions exist. Therefore, if $\mathbf{x}_3$ enters the basis and $\mathbf{x}_1$ leaves the basis, an alternative optimum simplex table is obtained as follows :

Table 5·59

		$c_j \to$	2	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
x_3	0	10	2	0	1	-1	
x_2	2	9	1	1	0	1	
		$z = 18$	0	0	0	2	$\leftarrow \Delta_j$

Thus optimum solution is : $x_1 = 0$, $x_2 = 9$.

(d) If c_2 becomes equal to 3 instead of 2, then there exists a unique optimal solution $x_1 = 0$, $x_2 = 9$.

(e) Inverse of optimal basis from the optimal table is read as

$$\mathbf{B}^{-1}=\begin{bmatrix}1/2 & -1/2\\ -1/2 & 3/2\end{bmatrix}$$

EXAMINATION PROBLEMS

1. Solve the following system of linear equations by simplex method.

   (i) $x_1 + x_2 = 1$
   $2x_1 + x_2 = 3$. **[Meerut M.Sc. (Maths.) 90]**
   [Ans. $x_1 = 2$, $x_2 = -1$**]**

   (ii) $3x_1 + 2x_2 = 4$
   $4x_1 - x_2 = 6$ **[Meerut M.Sc. (Maths.) 94]**
   [Ans. $x = 16/11$, $x_2 = -2/11$**]**

2. Find the inverse of the matrix

   (i) $\begin{bmatrix}4 & 1 & 2\\ 0 & 1 & 0\\ 8 & 4 & 5\end{bmatrix}$ (ii) $\begin{bmatrix}4 & 1\\ 2 & 9\end{bmatrix}$ **[Meerut (Maths.) 93]**

3. Consider the matrix $\mathbf{B} = (\beta_1 , \beta_2 , \beta_3)$ whose inverse is

   $$\mathbf{B}^{-1}=\begin{pmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ -1 & 2 & 1\end{pmatrix}$$

   Find the inverse matrix $\mathbf{B} = (\beta_1 , \beta_2 , \mathbf{e})$ where $\mathbf{e}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$.

3. Solve the system of equations by using simplex method :

   $3x_1 + 2x_2 + x_3 + 4x_4 \le 6$, $2x_1 + x_2 + 5x_3 + x_4 \le 4$, $2x_1 + 6x_2 - 4x_3 + 8x_4 = 0$,
   $x_1 + 3x_2 - 2x_3 + 4x_4 = 0$, and $x_1\ x_2$, $x_3\ x_4 \ge 0$. **[Meerut (Maths.) Jan. 98 (BP)]**

4. (a) In relation to linear programming, explain the implications of the following assumptions of the model :
   (i) Linearity of the objective function and constraints ; (ii) Continuous variables; (iii) Certainty.

   (b) An Air Force is experimenting with three types of bombs P, Q and R in which three kinds of explosives, viz. A, B and C will be used. Taking the various factors into consideration, it has been decided to use at most 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. Bomb P requires 3, 2, 2 kg of A, B and C respectively. Bomb Q requires 4, 3, 2 kg. of A, B and C. Bomb R requires 6, 2, 3 kg of A, B and C respectively. Now bomb P will give the equivalent of a 2-ton explosion, bomb Q will give a 3-ton explosion and bomb R will give a 4-ton explosion. Under what production schedule can the Air Force make the biggest bomb.

(c) Obtain the dual problem of the following L.P.P. : Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$, subject to the constraints : $5x_1 + 6x_2 - x_3 \le 3$, $-x_1 + x_2 + 3x_3 \ge 4$, $7x_1 - 2x_2 - x_3 \le 10$, $x_1 - 2x_2 + 5x_3 \ge 3$, $4x_1 + 7x_2 - 2x_3 \ge 2$, and $x_1, x_2, x_3 \ge 0$

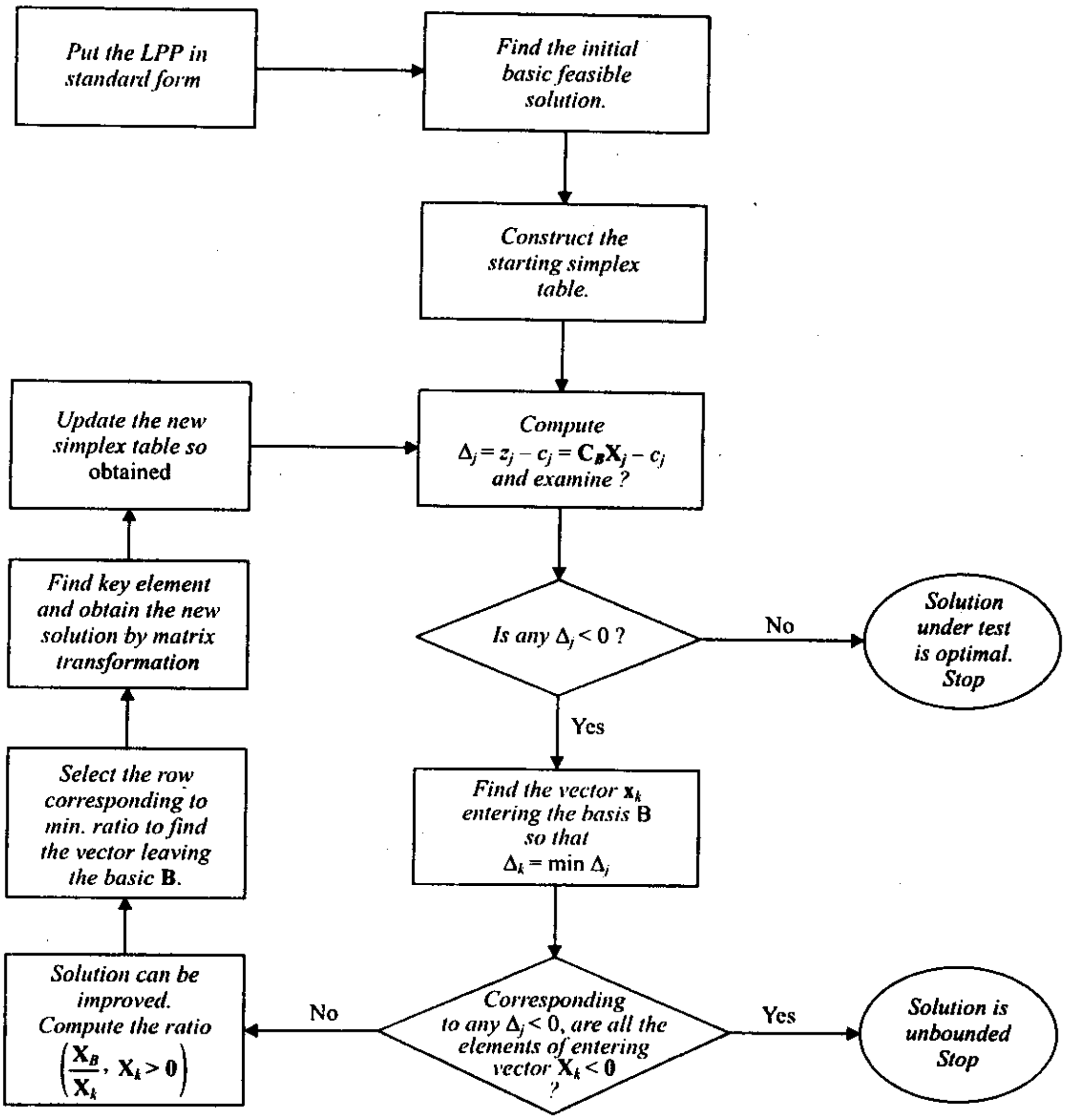
[ICWA (June) 91]

5·19 SUMMARY OF COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

Simplex method is an iterative procedure involving the following steps :

Step 1. If the problem is one of minimization, convert it to a maximization problem by considering $-z$, instead of z, using the fact $\min z = -\max(-z)$ or $\min z = -\max.(z')$, $z' = -z$.

FLOWCHART FOR SIMPLEX METHOD



Step 2. We check up all b_i's for nonnegativity. If some of the b_i's are negative, multiply the corresponding constraints through by -1 in order to ensure all $b_i \geq 0$.

Step 3. We change the inequalities to equations by adding slack and surplus variables, if necessary.

Step 4. We add artificial variables to those constraints with ($\geq$) or (=) sign in order to get the identity basis matrix.

Step 5. We now construct the starting simplex table (*see Table 5·60* for all problems). From this table, the initial basic feasible solution can be read off.

Table 5·60. Form of Simplex Table

		$c_j \rightarrow$	c_1	c_2	$c_3 \ldots c_k \ldots c_{m+n}$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3} \ldots \mathbf{X_k} \ldots \mathbf{X_{m+n}}$	MIN. RATIO RULE
...	...	...	...	...	... ...	...
	$z = \mathbf{C_B X_B}$		Δ_1	Δ_2	$\Delta_3 \ldots \Delta_k \ldots \Delta_{m+n}$	$\leftarrow \Delta_j$

Note. All the steps of simplex algorithm can be easily remembered by the **Flow-Chart** given below

Step 6. We obtain the values of Δ_j by the formula $\Delta_j = z_j - c_j = \mathbf{C_B X_j} - c_j$, and examine the values of Δ_j. There will be three mutually exclusive and collectively exhaustive possibilities :

(i) All $\Delta_j \geq 0$. In this case, the basic feasible solution under test will be optimal.

(ii) Some $\Delta_j < 0$, and for at least one of the corresponding $\mathbf{X}_j$ all $x_{rj} \leq 0$. In this case, the solution will be unbounded.

(iii) Some $\Delta_j \leq 0$, and all the corresponding $\mathbf{X}_j$'s have at least one $x_{ij} > 0$. In this case, there is no end of the road. So further improvement is possible.

Step 7. Further improvement is done by replacing one of the vectors at present in the basis matrix by that one out side the basis. We use the following rules to select such a vector :

(i) *To select "incoming vector". We find such value of k* for which $\Delta_k = \min \Delta_j$. Then the vector coming into the basis matrix will be $\mathbf{X_k}$.

(ii) *To Select "outgoing vector". The vector going out of the basis matrix will be* β_r, *if we determine the suffix r* by the minimum ratio rule

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right], \qquad \text{for predetermined value of } k.$$

Step 8. We now construct the next improvement table by using the simple matrix transformation rules.

Step 9. Now return to ***step 6,*** then go the ***steps 8*** and ***9,*** if necessary. This process is repeated till we reach the desired conclusion.

Q. 1. Give the outlines of simplex method of linear programming. **[Banasthali (M.Sc.) 93]**

2. Show how simplex method can be applied to find a solution of the following system : $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq 0$.

3. Give a flow chart of the simplex method. **[Kanpur (B.Sc.) 91]**

SELF EXAMINATION PROBLEMS

1. Solve the following problems by simplex method adding aftificial variables :

Max. $z = 2x_1 + 5x_2 + 7x_3$,
subject to
$3x_1 + 2x_2 + 4x_3 \leq 100$
$x_1 + 4x_2 + 2x_3 \leq 100$
$x_1 + x_2 + 3x_3 \leq 100$
$x_1, x_2, x_3 \geq 0$
[Ans. $x_1 = 0$, $x_2 = 50/3$, $x_3 = 50/3$, $\max z = 200$]

2. Max. $z = 4x_1 + x_2 + 4x_3 + 5x_4$, subject to the constraints :
$4x_1 + 6x_2 - 5x_3 + 4x_4 \geq -20$, $3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$, $8x_1 - 3x_2 - 3x_3 + 2x_4 \leq 20$, and $x_1, x_2, x_3, x_4 \geq 0$.
[Ans. Unbounded solution]. **[Agra 98]**

3. A manufacturer has two products P_1 and P_2, both of them are produced in two steps by machines M_1 and M_2. The process times per hundred for the products on the machines are :

	M_1	M_2	Contribution (per hundred)
P_1	4	5	10
P_2	5	2	5
Available (hrs)	100	80	

The manufacturer is in a market upswing and can sell as much as he can produce of both products. Formulate the mathematical model and determine optimum product mix using simplex method.
[**Hint.** Formulation of the problem is : Max. $z = 10x_1 + 5x_2$, subject to $4x_1 + 5x_2 \le 100$, $5x_1 + 2x_2 \le 80$, x_1 , $x_2 \ge 0$] .
[**Ans.** $x_1 = 20000/17 \approx 1177$, $x_2 = 18000/17 \approx 1059$].

4. An animal feed company must produce 200 kgs of a mixture consisting of ingredient X_1 and X_2 daily. X_1 costs Rs. 3 per kg and X_2 Rs. 8 per kg. No more than 80 kgs of X_1 can be used and at least 60 kgs of X_2 must be used.Find how much of each ingredient should be used if the company wants to minimize cost.
[**Hint.** Formulation of the problem is : Min. $3x_1 + 8x_2$, subject to $x_1 + x_2 = 200$, $x_1 \le 80$, $x_2 \ge 60$, x_1 , $x_2 \ge 0$,
Substitute $x_1 = X_1 + 80$ in the problem and then solve by simplex method.
[**Ans.** $x_1 = 80$, $x_2 = 120$, min cost = Rs. 1200].

5. A company produces three products A , B and C . These products require three ores O_1 , O_2 and O_3. The maximum quantities of the ores O_1 , O_2 and O_3 available are 22 tons, 14 tons and 14 tons respectively. For one ton of each of these products, the ore requirements are :

	A	B	C
O_1	3	—	3
O_2	1	2	3
O_3	3	2	0
Profit per ton (Rs. in thousand)	1	4	5

The company makes a profit of one, four and five thousands on each ton of the products A , B and C respectively. How many tons of each products A , B and C should the company produce to maximize the profits.
[**Hint.** Formulation of the problem is : Max. $z = x_1 + 4x_2 + 5x_3$,
subject to $3x_1 + 3x_3 \le 22$, $x_1 + 2x_2 + 3x_3 \le 14$; $3x_1 + 2x_2 \le 14$; x_1 , x_2 , $x_3 \ge 0$.]

6. A furniture company manufactures four models of desks. Each desk is first constructed in the carpentry shop and is next sent to the finishing shop where it is varnished, waxed and polished. The number of man-hours of labour required in each shop is as follows :

Shop	Desk			
	I	II	III	IV
Carpentry	4	9	7	10
Finishing	1	1	3	40
Profit per item (Rs.)	12	20	18	40

Because of limitation in capacity of the plant, not more than 6,000 man-hours can be expected in the carpentry shop and 4,000 in the finishing shop in a month. Assuming that raw materials are available in adequate supply and all desks produced can be sold, determine the quantities of each type of desk to be made for maximum profit of the company.
[**Hint.** Formulation of the problem is : Max. $z = 12x_1 + 20x_2 + 18x_3 + 40x_4$,
subject to $4x_1 + 9x_2 + 7x_3 + 10x_4 \le 6{,}000$, $x_1 + x_2 + 3x_3 + 40x_4 \le 400$; x_1 , x_2 , x_3 , $x_4 \ge 0$.
[**Ans.** $x_1 = 4000/3$, $x_2 = x_3 = 0$, $x_4 = 200/3$, and max. $z = Rs.\ 56000/3$]

7. A factory has decided to diversify their activities, and data collected by sales and production is summerized below :
Potential demand exists for 3 products A , B and C . Market can take any amount of A and C whearas the share of B for this organization is expected to be not more than 400 units a month.
For every three units of C produced, there will be one unit of a by-product which sells at a contribution of Rs. 3 per unit, and only 100 units of this by-product can be sold per month. Contribution per unit of products A , B and C is expected to be Rs. 6, Rs. 8 and Rs. 4 respectively.
These products require 3 different processes, and time required per unit production is given in the following table :

Process	Product (hrs/unit)			Available
	A	B	C	
I	2	3	1	900
II	—	1	2	600
III	3	2	2	1200

Determine the optimum product-mix for maximizing the contribution.

[**Hint.** The formulation of the problem is : Max. $z = 6x_1 + 8x_2 + 4x_3 + 3x_4$, subject to
$2x_1 + 3x_2 + x_3 \le 900$, $3x_1 + 2x_2 + 2x_3 \le 1200$, $3x_3 + x_4 = 100$; $x_2, x_3 \ge 0$],
[Ans. $x_1 = 360$, $x_2 = 60$, $x_3 = 0$, $x_4 = 100$; max z = Rs. 2940]

8. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans costs Rs. 70 to prepare requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8000 may-days of work, how many acres should be allocated to each crop to maximize profit ?
[**Hint.** Formulation of the problem is : Max $z = 30x_1 + 40x_2 + 20x_3$, subject to
$10x_1 + 12x_2 + 7x_3 \le 10{,}000$, $7x_1 + 10x_2 + 8x_3 \le 8000$, $x_1 + x_2 + x_3 \le 1000$, and $x, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 250$, $x_2 = 625$, $x_3 = 0$; max z = Rs. 32,500].

9. A transistor radio company manufactures four models A, B, C and D which have profit contributions of Rs. 8, Rs. 15 and Rs. 25 on models A, B and C respectively and a loss of Re. 1 on model D. Each type of radio requires a certain amount of time for the manufacturing of components for assembling and for packing. Specially a dozen units of model A require one hour of manfacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2 and for a dozen units of C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only, During the forthcoming week, the comapny will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Obtain the optimal production schedule for the company.
[**Hint.** Formulation of the problem is : max. $z = 8x_1 + 15x_2 + 25x_3 - x_4$, subject to
$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$; $x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, max z = Rs. 120].

10. A manufacturing firm has discountinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering to devote this excess capacity to one or more of three products : call them product 1, 2 and 3. The available capacity on the machines which might limit output, is summarized in the following table.

Machine Type	Available Time (in machine hours per week)
Milling Machine	250
Lathe	150
Grinder	50

The number of machine-hours required for the unit of the respective product is given below :

Machine Type	Productivity 33 (in machine-hours per unit)		
	Product 1	Product 2	Product 3
Milling Machine	8	2	3
Lathe	4	3	0
Grinder	2	—	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for products 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit ?
[**Hint.** Formulation of the problem is : Max. $z = 20x_1 + 6x_2 + 8x_3$, subject to
$8x_1 + 2x_2 + 3x_3 \le 250$; $4x_1 + 3x_2 \le 150$, $2x_1 + x_3 \le 50$, and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 50$, $x_3 = 50$; max z = 700].

11. The XYZ company manufactures two products A and B. These products are processed on the same machine. It takes 25 minutes to process one unit of product A and 15 minutes for each unit of product B and the machine operates for a maximum of 35 hours in a week. product A requires 1 kg. of the raw material per unit, the supply of which is 170 kgs. per week.

If the net income from the products are Rs. 100 and Rs. 450 per unit respectively and manufacturing costs are proportional to the square of the quantity made for each product, find how much of each product should be produced per week, in order to maximize profits.

[**Ans.** 68 units of B, no unit of A, max profit = Rs. 30,600].

12. A manufacturer of steel furniture makes three products—Chairs, Filing cabinetes and Tables. Three machines (call them A, B and C) are available on which these products are processed. The manufacturer has 100 hours per week available on each of three machines. The time required by each of the three products on three machines is summarized in the following table :

Product	Time required (in hours)		
	Machine A	Machine B	Machine C
Chair	2	2	1
Filing cabinet	2	1	2
Table	—	1	2

The profit analysis shows that the net profit on each chair, filing–cabinet and table is Rs. 22, Rs. 30 and Rs. 25 respectively.

What should be the weekly production of these products so that the manufacturer's total profit per week is maximized.
[**Ans.** 100/3 chairs, 50/3 tables, and 50/3 file cabintes. max. profit = Rs. 1650]

13. A manufacture produces three products *A*, *B* and *C*. Each product can be produced on either one of two machines, I and II. The time required to produce 1 unit of each product on a machine is given in the table below :

Product	Time to Produce 1 unit (hours)	
	Machine I	Machine II
A	0·5	0·6
B	0·7	0·8
C	0·9	1·05

There are 85 hours available on each machine ; the operating cost is Rs. 5 per hour for machine I and Rs. 4 per hour for machine II and the product requirements are at least 90 units of *A*, at least 80 units *B*, and at least 60 units of *C*. The manufacturer wishes to meet the requirements at minimum cost.

Solve the given linear programming problem by simplex method.
[**Ans.** 150 hrs. on machine II, no time on machine I, min cost = Rs. 600]

14. A Plant is engaged on the production of two products which are processed through three departments, the number of hours required to finish each is indicated in the table below :

Department	Product		Max. hours available per week
	A	*B*	
I	7	8	1600
II	8	12	1600
III	15	16	1600

(a) If the profit for the products is Rs. 6 for a unit of product *A* but only Rs. 4 for a unit of product. *B*, what quantities per week should be planned to maximize profit. Illustrate the problem graphically.
(b) Capacity can be increased on one department only. In which department should it be done and why ? To what extent should the capacity be increased ?
(c) If the cost per hour in department I is Rs. 25 ; in department II, Rs. 40; and in department III, Rs. 50 ; what quantities should be planned to minimize the cost of production ?

[**Ans.** (a) 320/3 units of product *A* and no unit of product *B*; max. profit is Rs. 640.
(c) 10/3 units of product *A* and no unit of product *B*; min. cost = Rs. 20].

15. A television comapny has three major departments for manufacture of its models *A* and *B*. Monthly capacities are given as follows :

Department	Per Unit Time Requirement (Hrs.)		Hrs. available this month
	Model *A*	Model *B*	
I	4·0	2·0	1,600
II	2·5	1·0	1,200
II	4·5	1·5	1,600

The marginal prifit of model *A* is Rs. 400 each and that of model *B* is Rs. 100 each. Assuming that the company can sell any quantity of either product due to favourable market conditions; determine the optimum output for both the models, the highest possible profit for this month and the slack time in the three departments.

[**Hint.** Formulation is : Max. $z = 400x_1 + 100x_2$, subject to the conditions :

$4x_1 + 2x_2 \le 1600;\ 5/2\ x_1 + x_2 \le 1200\ ;\ 9/2\ x_1 + 3/2\ x_2 \le 1600\ ;\ x_1 \ge 0\ ,\ x_2 \ge 0$]

[**Ans.** $x_1 = 3200/9$, $x_2 = 0$, $x_3 = 0$, max. $z = 1280000/9$]

16. In the solution of linear programming problems by Simplex Method, for deciding the leaving variable :
(a) The maximum negative coefficient in the objective function row is selected.
(b) The minimum positive ratio of the right hand side to the first decision variable is selected.
(c) The maximum positive ratio of the right hand side to the coefficients in the key column is selected.
(d) The minimum positive ratio of the right hand side to the coefficient in the key column in selected. **[IES 2003]**

SELF–EXAMINATION QUESTIONS

1. Establish the difference between (i) feasible solution, (ii) Basic feasible solution and (iii) degenerate basic feasible solution.

2. (a) Define a basic solution to a given system of *m* simultaneous linear equations in *n* unknowns.
(b) How many basic feasible solutions are there to a given system of 3 simultaneous linear equations in 4 unknown.

3. Define the following terms :
(i) basic variable (ii) basic solution (iii) basic feasible solution (iv) degenerate solution.

4. Give outlines of simplex method in linear programming. Why is it so called.
5. What do you mean by two phase method for solving a given L.P.P. Why is it used.
6. What are the various methods known to you for solving a linear programming problem ?
7. What is the pivoting process ?
8. Name the three basic parts of the simplex technique.
9. Give the geometric interpretation of the simplex procedure.
10. Write the role of pivot element in a simplex table. **[Madurai B.Sc. (Com. Sc.) 92]**
11. In the course of simplex table calculations, describe how you will detect a degenerate, an unbounded and a non existing feasible solution.
12. What is degeneracy in simplex ? Solve the following LP problem using simplex : Max. $z = 3x_1 + 9x_2$, s.t. $4x_1 + 4x_2 \le 8$, $x_1 + 2x_2 \le 4$ and $x_1, x_2 \ge 0$. **[IPM (PGDBM) 2000]**
13. With reference to the solution of LPP by simplex method/table when do you conclude as follows : (i) LPP has multiple solutions, (ii) LPP has no limit for the improvement of the objective function, (iii) LPP has no feasible solution. **[VTU (BE Mech.) 2002]**

MODEL OBJECTIVE QUESTIONS
(ON THEORY OF SIMPLEX METHOD)

1. Fill-in the blanks so that the following statements are correct. Write only the answers.
   (i) The evaluation of z_j is given by ....
   (ii) The vector a_k (in the simplex table) for which $\Delta_k = \min \Delta_j$ (where $\Delta_j = z_j - c_j$) is called the ....
   (iii) The vector β_r (in the simplex table) for which
   $$\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}},\ y_{ik} > 0 \right\}$$
   is called the ....
   (iv) While solving an LPP by 'Big-M' method, there are only two negative values of Δ_j (*i.e.*, $z_j - c_j$), namely, $5M + 10$ and $6M - 1$. Then the incoming vector true will correspond to the value ....
2. Each of the following statements is either true/ or false. Indicate your choice of the answer by writing TRUE or FALSE for each statement.
   (i) If an LPP has an optimal solution, then at least one BFS must be optimal.
   (ii) If an LPP has a feasible solution, then it is not necessary that it has a BFS.
   (iii) If a column vector α_j belongs to the basis matrix, then $z_j - c_j \neq 0$.
   (iv) It is always possible to find an initial BFS to an LPP.
   (v) The following LPP cannot be solved by simplex method unless artificial variables are introduced :
   Minimize $z = 4x_1 + 8x_2 + 3x_3$
   subject to the constraints
   $x_1 + x_2 \ge 2$, $2x_1 + x_3 \ge 5$, and $x_1, x_2, x_3 \ge 0$.
   (vi) The following LPP has an unbounded optimal solution :
   Maximize $z = 2x_1 + x_2$
   subject to the constraints :
   $x_1 - x_2 \le 10$, $2x_1 - x_2 \le 40$, and $x_1, x_2 \ge 0$.
   (vii) An LPP has unbounded feasible region if all the elements of some column vector in the starting simplex table are non-positive.
   (viii) A system of simultaneous linear equations may be solved by simplex method by introducing a dummy objective function with price zero to each given variable and price -1 to each artificial variable.
   (ix) If α_2 is the incoming vector and β_3 is the outgoing vector at some iteration of the simples method, then the key element will be y_{23}.
   (x) A BFS to an LPP is optimal if $z_j - c_j \ge 0$ for all j for which the column vector $\alpha_j \in A$ is not in the basis.
3. For each of the following statements one of the four alternatives is correct. Indicate your choice of correct answer for each statement by writing one of the letters a, b, c, d whichever is appropriate.
   (i) An LPP has an unbounded optimal solution if there is some non-basis vector α_j for which
   (a) $z_j - c_j < 0$ and $y_{ij} > 0$ for all i (b) $z_j - c_j < 0$ and $y_{ij} \le 0$ for all i
   (c) $z_j - c_j > 0$ and $y_{ij} \le 0$ for all i (d) $z_j - c_j < 0$ and $y_{ij} \ge 0$ for all i
   (ii) If there is an optimal BFS to an LPP, then an alternative non-basic optimal solution exists if for some non-basis vector α_j we have

(a) $z_j - c_j = 0$ and $y_{ij} > 0$ for all i
(b) $z_j - c_j = 0$ and $y_{ij} < 0$ for all i
(c) $z_j - c_j < 0$ and $y_{ij} < 0$ for all i
(d) $z_j - c_j > 0$ and $y_{ij} > 0$ for all i.

(iii) If there is an optimal BFS to an LPP, then an alternative basic optimal solution exists if for some non-basic vector α_j we have
(a) $z_j - c_j = 0$ and $y_{ij} > 0$ for all i (b) $z_j - c_j = 0$ and $y_{ij} < 0$ for all i
(c) $z_j - c_j < 0$ and $y_{ij} < 0$ for all i (d) $z_j - c_j > 0$ and $y_{ij} > 0$ for all i

(iv) In a simplex table with usual notations, where $\Delta_j = c_j - z_j$, the incoming vector α_k is determined by
(a) $\Delta_k = \min(-\Delta_j)$ (b) $\Delta_k = \min \Delta_j$
(c) $\Delta_k = \max(-\Delta_j)$ (d) $\Delta_k = \max \Delta_j$.

(v) If a simplex table with usual notations, where $\Delta_j = z_j - c_j$, the outgoing vector β_r is determined by
(a) $\dfrac{x_{Br}}{y_{rk}} = \min\limits_i \left\{ \dfrac{x_{Bi}}{y_{ik}}, y_{ik} < 0 \right\}$ (b) $\dfrac{x_{Br}}{y_{rk}} = \max\limits_i \left\{ \dfrac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right\}$
(c) $\dfrac{x_{Br}}{y_{rk}} = \max\limits_i \left\{ \dfrac{x_{Bi}}{y_{ik}}, y_{ik} < 0 \right\}$ (d) $\dfrac{x_{Br}}{y_{rk}} = \min\limits_i \left\{ \dfrac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right\}$.

Answers

1. (i) $c_B y_j$ (ii) Incoming vector (iii) Outgoing vector (iv) $6M + 1$
2. (i) True (ii) False (iii) False (iv) True (v) False (vi) True (vii) True (viii) True (ix) False (x) True.
3. (i) c (ii) b (iii) a (iv) b (v) d.

TRUE OR FALSE QUESTIONS

State which of the following statements are True or False :

1. Each constraint in the standard equality form model (excluding non-negativity conditions) is represented by a row of the simplex tableau. (T F)
2. For a max. LP model, it is the extra rule that guarantees a non-decreasing objective function in each move. (T F)
3. In simplex methd, all variables must be non-negative. (T F)
4. In the simplex method, any variable which is removed from the basis at an iteration cannot re-enter the basis at the next iteration. (T F)
5. Constraints involving 'equal to' sign do not require use of slack or artificial variables. (T F)
6. If all the constraints of a maximization LP problem are of 'greater than or equal to' type, the problem shall have no feasible solution. (T F)
7. In a simplex method, the pivot number can be zero or negative. (T F)
8. In solving a minimum model, opposite to a maximum model, the entering rule and optimally criterion are the only differences. (T F)
9. In the simplex method, the optimally conditions for the maximization and minimization problems are different. (T F)
10. In the simplex method, the feasibility condition for the maximization and minimization problems are different. (T F)
11. For a maximization problem, the pivot column refers to the column with the lowest negative index number. (T F)
12. An optimal solution results when all index numbers are non-negative for maximization problem, and are non-positive for minimization problem. (T F)
13. If in the final simplex tableau of a linear programming problem, any basic variable has a value of zero in the quantity column, the problem has multiple optimal solutions. (T F)
14. All the rules and procedures of the simplex method are identical whether solving a maximization or a minimization problem. (T F)
15. Every feasible solution obtained by the simplex method corresponds, to an extreme point of the feasible region of the graphical method. (T F)
16. If an optimal solution is degenerate, it is measningless to the manager.
17. Degeneracy will result whenever a tie occurs in the minimum ratio for the entering rule. (T F)
18. If a current iteration is degenerate, the next iteration will necessarily be degenerate. (T F)
19. An LP problem may have a feasible solution even through an artificial variable appears at a positive level in the final iteration. (T F)
20. In the simplex method, the volume of computations increases primarily with the number of constraints. (T F)
21. The optimally condition always guarantees that the next solution will have an improved value of the objective function than in the immediately preceding iteration. (T F)

22. Every feasible point in a bounded LP solution can be determined from its feasible extreme points. (T F)
23. An artificial variable column can be dropped all together from the simplex tableau once the variable becomes non-basic. (T F)
24. The two-phase method and the big *M*-method require the same number of iterations for solving a linear program. (T F)
25. Degeneracy can be avoided if redundant constraints can be deleted. (T F)
26. The simplex method may not move to an adjacent extreme point if the current iteration is degenerate. (T F)
27. If the solution space is unbounded, the objective value always will be unbounded. (T F)

Answers

1. T	2. T	3. T	4. F	5. F	6. F	7. F	8. T	9. T
10. F	11. F	12. T	13. F	14. F	15. T	16. F	17. F	18. F
19. F	20. F	21. F	22. T	23. T	24. T	25. T	26. T	27. F.

MULTIPLE CHOICE QUESTIONS

Out of the following multiple choice questions, indicates the appropriate answer :

1. When making a pivot in the simplex method, the inter-sectional elements are always found in the
   (a) $z_j - c_j$ row. (b) Optimal column.
   (c) Quantity column. (d) None of the above.
2. Suppose that one of the substitution rates in a simplex tableaue is negative. This implies that
   (a) Adding one unit of the variable heading that column to the production mix would result in a possible increase in the number of units in the production mix for the quantity corresponding to that row.
   (b) Adding one unit of the variable heading that column to the production mix would decrease the quantity of the row variable in the production mix.
   (c) The variable corresponding to that now will not leave the solution on this iteration.
   (d) Both (a) and (c).
3. Every simplex iteration for a minimization problem replaces a variable in the current basis with another variable which has
   (a) a larger per unit profitability as shown in the c_j (*i.e.*, objective function coefficient) row.
   (b) a negative $z_j - c_j$ value.
   (c) the greatest $c_j - z_j$ value.
   (d) any positive $z_j - c_j$ value.
4. Every taleau in the simplex method
   (a) exhibits a solution to the original equations.
   (b) exhibits a basic feasible solution to the equation in the standard equality form of the model.
   (c) corresponds to an extreme point of the constraint set.
   (d) exhibits a set of transformed equations.
   (e) all of the above.
5. The signal for optimality in a max. model is
   (a) $z_j - c_j \le 0$, for all j. (b) $z_j \le 0$, for all j. (c) $z_j - c_j \ge 0$, for all j.
6. Artificial variables
   (a) are used to aid in finding an initial solution.
   (b) are used in phase 1 of two phase method.
   (c) can be used to find optimal dual prices in the final tableau.
   (d) all of the above.
7. Which of the following is not true of the simplex method ?
   (a) At each iteration, the objective value either stays the same or improves.
   (b) It indicates an unbounded or infeasible problem.
   (c) It signals optimality.
   (d) It coverges in at most m steps, where m is the number of constraints.
8. Infeasibility is discovered
   (a) in computing the entering variable.
   (b) in computing the departing variable.
   (c) none of the above.
9. Suppose that in a non-degenerate optimal tableau, a slack variable s_2 is basic for the second constraint, whose RHS is b_2. This means that
   (a) the original problem is infeasible.
   (b) all of b_2 is used up in the optimal solution.
   (c) both the dual price and the optimal value of the dual variable, for the second constraint, are zero.
   (d) a better optimal value could be obtained by increasing b_2.

10. The simplex method has the property that
(a) At each iteration it gives a solution which is at least as good as the earlier solution.
(b) At each stage it produces feasible solution.
(c) It signals that optimal solution which has been found. (d) None of the above.
11. Cycling
(a) can always be prevented. (b) is a real-world concern.
(c) will cause more pivots to occur before termination.
12. How is the variable which will be replaced in the next simplex solution determined ?
(a) By choosing the variable with the smallest $z_j - c_j$ value.
(b) By choosing the variable with the largest $c_j - z_j$ value.
(c) By choosing the variable which yield the minimum positive ratio of quantity to substitution rate in the optimal column.
(d) It can be chosen arbitrarily.

MODEL OBJECTIVE QUESTIONS
(ON APPLICATIONS)

13. The role of artificial variables in the simplex method is
(a) to aid in finding an initial solution. (b) to find optimal dual prices in the final simplex table.
(c) to start phases of simplex method. (d) all of the above.
14. For a maximization problem, the objective function coefficient for an artificial variable is
(a) + M. (b) − M. (c) zero. (d) none of the above.
15. If a negative value appears in the solution values (x_B) column of the simplex table, then
(a) the solution is optimal. (b) the solution is infeasible.
(c) the solution is unbounded. (d) all of the above.
16. At every iteration of simplex method, for minimization problem, a variable in the current basis is replaced with another variable that has
(a) a negative $z_j - c_j$ value. (b) a positive $z_j - c_j$ value. (c) $z_j - c_j = 0$. (d) none of the above.
17. In the optimal simplex table, $z_j - c_j = 0$ indicates
(a) unbounded solution. (b) cycling. (c) alternative solution. (d) infeasible solution.
18. For maximization LP model, the simplex method is terminated when all values
(a) $z_j - c_j \geq 0$. (b) $z_j - c_j \leq 0$. (c) $z_j - c_j = 0$. (d) $z_j \leq 0$.
19. A variable which does not appear in the basic variable (B) column of simplex table is
(a) never equal to zero. (b) always equal to zero. (c) called a basic variable. (d) none of the above.
20. If for a given solution, a slack variable is equal to zero, then
(a) the solution is optimal. (b) the solution is infeasible.
(c) the entire amount of resource with the constraint in which the slack variable appears has been consumed.
(d) all of the above.
21. If an optimal solution is degenerate, then
(a) there are alternative optimal solutions. (b) the solution is infeasible.
(c) the solution is of no use to the decision-maker. (d) none of the above.
22. To formulate a problem for solution by the simplex method, we must add artificial variable to
(a) only equality constraints. (b) only 'greater than' constraints.
(c) both (a) and (b). (d) none of the above.
23. A simplex table for a linear programming problem is given below :

	$c_j \rightarrow$ 5	2	3	0	0	0	
Basic Var.	X_1	X_2	X_3	X_4	X_5	X_6	Z
x_4	1	2	2	1	0	0	8
x_5	3	4	1	0	1	0	7
x_6	2	3	4	0	0	1	10

Which one of the following correctly indicates the combination of entering and leaving variables ?
(a) x_1 and x_4. (b) x_2 and x_6. (c) x_2 and x_5. (d) x_3 and x_4.
[IES (Mech.) 1994]
24. Which one of the following subroutines does a computer implimentation in linear programming by the simplex method use ?
(a) Finding a root of a polynomial. (b) Finding the determinant of a matrix.
(c) Finding the eigen values of a matrix. (d) Solving a system of linear equations.
[IES (Mech.) 1996]

25. Consider the following statements :

1. A linear programming problem with three variables and two constraints can be solved by graphical methd.
2. For solutions of a linear programming problem with mixed constraints, Big-M method can be employed.
3. In the solution process of a linear programming problem using Big-M method when an artificial variable leaves the basis, the column of the artificial variable can be removed from all subsequent tables.

Which of these statements are correct ?

(a) 1, 2 and 3. (b) 1 and 2. (c) 1 and 3. (d) 2 and 3.

[IES (Mech.) 2000]

26. Consider the following statements regarding linear programming :

1. Dual of a dual is primal.
2. When two minimum ratios of the right hand side to the coefficient in the key column are equal, degeneacy may take place.
3. When an artificial variable leaves the basis, its column can be deleted from the subsequent simplex tables.

Select the correct answer from the codes given below :

codes :

(a) 1, 2 and 3. (b) 1 and 2. (c) 2 and 3. (d) 1 and 3.

[IES (Mech.) 2001]

27. In the solution linear programming problems by Simplex Method, for deciding the leaving variable

(a) The maximum negative coefficient in the objective function row is selected.
(b) The minimum positive ratio of the right hand side to the first decision variable is selected.
(c) the maximum positive ratio of the right hand side to the coefficients in the key column is selected.
(d) The minimum positive ratio of the right hand side to the coefficient in the key column is selected. **[IES 2003]**

Answers

1. (b)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (c)
10. (d)	**11.** (c)	**12.** (c)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (a)
19. (b)	**20.** (c)	**21.** (a)	**22.** (d)	**23.** (c)	**24.** (d)	**25.** (d)	**26.** (a)	**27.** (d).

6

Transportation Model

6.1 INTRODUCTION

As already defined and discussed earlier, the simplex procedure can be regarded as the most generalized method for linear programming problems. However, there is very interesting class of *'Allocation Methods'* which is applied to a lot of very practical problems generally called *'Transportation Problems'*. Whenever it is possible to place the given linear programming problem in the transportation frame-work, it is far more simple to solve it by *'Transportation Technique'* than by *'Simplex'*.

Let the nature of transportation problem be examined first. If there are more than one centres, called *'origins'*, from where the goods need to be shipped to more than one places called *'destinations'* and the costs of shipping from each of the *origins* to each of the *destinations* being different and known, the problem is to ship the goods from various *origins* to different *destinations* in such a manner that the cost of shipping or transportation is minimum.

Thus, we can formally define the transportation problem as follows :

Definition. ***The Transportation Problem** is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.*

For example, a tyre manufacturing concern has m factories located in m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

The various features of linear programming can be observed in these problems. Here the availability as well as the requirements of the various centres are finite and constitute the limited resources. It is also assumed that the cost of shipping is linear (for example, the costs of shipping of *two* objects will be *twice* that of shipping a *single* object). However, this condition is not often true in practical problems, but will have to be assumed so that the linear programming technique may be applicable to such problems. These problems thus could also be solved by 'Simplex'. Mathematically, the problem may be stated as given in the following section.

6.2 MATHEMATICAL FORMULATION

Let there be m origins, ith origin possessing a_i units of a certain product, whereas there are n destinations (n may or may not be equal to m) with destination j requiring b_j units. Costs of shipping of an item from each of m origins (sources) to each of the n destinations are known either directly or indirectly in terms of mileage, shipping hours, etc. Let c_{ij} be the cost of shipping one unit product from ith origin (source) to jth destination., and 'x_{ij}' be the amount to be shipped from ith origin to jth destination.

It is also assumed that total availabilities Σa_i satisfy the total requirements Σb_j, *i.e.*,

$$\Sigma a_i = \Sigma b_j \quad (i = 1, 2, \ldots, m \,;\, j = 1, 2, \ldots, n) \qquad \ldots(6\cdot1)$$

(***In case,*** $\Sigma a_i \neq \Sigma b_j$ some manipulation is required to make $\Sigma a_i = \Sigma b_j$, which will be shown later).

The problem now is to determine non-negative (≥ 0) values of 'x_{ij}' satisfying both, the availability constraints :